CSE 312 Midterm – Taylor Blau

1 Combinatorics

1.1 Sum & Product Rule

The product rule states that for distinct events $A_1, \ldots, A_n$ each having $m_1, \ldots, m_n$ outcomes overall, the total number of outcomes is given as: $\prod_i m_i$.

The sum rule states that for the same scenario, where $A_1, \ldots, A_n$ are independent, that the total number of outcomes is instead given as: $\sum_i m_i$.

1.2 Ordering

The number of ways to order n distinct objects is $n!$. The number of ways to *permute* n objects (arrange a k-subset where order matters) is given as:

$$P(n,k) = \frac{n!}{(n-k)!}$$

Similarly, the number of ways to *combine* n objects (form a k-subset where order does not matter) is:

$$C(n,k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

1.3 Multinomial Coefficient

The number of ways to distinctly arrange n objects where $k < n$ of them are distinct is given as (where n_i represents the number of occurrences of the i-th distinct element) :

$$\frac{n!}{\prod_i n_i!} = \binom{n}{n_1, \ldots, n_k}$$

1.4 Pigeonhole Principle

If there are n pigeons and k holes where $n > k$, then there exists a hole with at least $\lceil n/k \rceil$ pigeons.

1.5 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},\ x, y \in \mathbb{R}, n \in \mathbb{N}$$

1.6 Principle of Inclusion-Exclusion

Suppose A and B are non-distinct set (i.e., $A \cap B \neq \emptyset$), then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

And so on for more sets, alternating adding and subtracting different size pairs.

1.7 Stars and Bars

For r bins and n objects:

Distinct objects, distinct bins r^n.

k-distinct objects, distinct bins $\binom{n}{k} r^k$.

Identical objects, distinct bins $\binom{n+r-1}{r-1}$.

Identical objects, non-empty distinct bins $\binom{n-1}{r-1}$.

2 Probability

2.1 Complementation

When an event E is difficult to count, but its complement $\overline{E}$ is easier, the following applies:

$$\mathbb{P}[E] = 1 - \mathbb{P}[\overline{E}]$$

This is useful in solving statements like "contains at least n", where the following is applied:

$$\mathbb{P}[X \geq N] = 1 - \mathbb{P}[X < N] = 1 - \sum_{i=1}^{n-1} \mathbb{P}[X = i]$$

2.2 Partitioning

Non-empty events E_i partition Ω iff:

1. E is *exhausted*, i.e., $\bigcup_i E_i = \Omega$, and
2. E_i (for $i \in [n]$) is *pairwise mutually exclusive*, i.e., $\forall i \neq j.\ E_i \cap E_j = \emptyset$.

2.3 Miscellanea

1. $\mathbb{P}[E] \geq 0$
2. $\mathbb{P}[\Omega] = 1$
3. $E \perp\!\!\!\perp F \to \mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F]$
4. $\mathbb{P}[E] + \mathbb{P}[E^C] = 1$
5. If $E \subseteq F$, then $\mathbb{P}[E] \leq \mathbb{P}[F]$

2.4 Equally Likely Outcomes

When every outcome in a (finite) sample space Ω is equally likely, and $E \subseteq \Omega$ is an event, then: $\mathbb{P}[E] = |E|/|\Omega|$.

2.5 Conditional Probability

Probability of an event can be *conditioned* on another event, i.e., we compute the probability of E *given* F. This is denoted:

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

We say that events E and F are *independent* iff

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F] \quad \text{or,}$$
$$\mathbb{P}[E] = \mathbb{P}[E \mid F]$$

2.6 Law of Total Probability

Suppose that $A_1, \ldots, A_n$ is a partition of Ω, and $E \subseteq \Omega$. Then:

$$\mathbb{P}[E] = \sum_{i=1}^{n} \mathbb{P}[E \mid A_i] \cdot \mathbb{P}[A_i]$$

2.7 Bayes' Theorem (& Total Probability)

$$\mathbb{P}[E \mid F] = \frac{\mathbb{P}[F \mid E] \cdot \mathbb{P}[E]}{\mathbb{P}[F]} = \frac{\mathbb{P}[F \mid E] \cdot \mathbb{P}[E]}{\mathbb{P}[F \mid E] \cdot \mathbb{P}[E] + \mathbb{P}[F \mid E^C] \cdot \mathbb{P}[E^C]}$$

2.8 Chain Rule

Suppose that $A_1, \ldots, A_n$ are events. Then:

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} \mathbb{P}\left[A_i \,\middle|\, \bigcup_{j=1}^{(i-1)} A_j\right]$$

3 Random Variables

A *random variable (r.v.)* is a function $X : \Omega \to \mathbb{R}$ that maps events to numeric values. We say that the *range* of X is denoted Ω_X.

3.1 Probability Functions

If X is discrete, then its probability mass function (where $p_X : \Omega_X \to [0,1]$) is given:

$$p_X(k) = \mathbb{P}[X = k] \quad \text{and} \quad \sum_x p_X(x) = 1$$

Further, if X is discrete, then its *cumulative distribution function* is defined as $F : \mathbb{R} \to [0,1]$, and is given as:

$$F_X(k) = \mathbb{P}[X \leq k] = \sum_{k' \in \Omega_X}^{k} \mathbb{P}[X = k']$$

3.2 Expectation

The *expectation* of a random variable is denoted as $\mathbb{E}[X] = \mu_X$, and is given as:

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot p_X(x) = \sum_{x \in \Omega_X} x \cdot \mathbb{P}[X = x]$$

There are several important axioms of expectation:

1. $\mathbb{E}[c] = c,\ c \in \mathbb{R}$
2. $\mathbb{E}[cX] = c \cdot \mathbb{E}[X],\ c \in \mathbb{R}$
3. $\mathbb{E}[aX + bY] = a \cdot \mathbb{E}[X] + b \cdot \mathbb{E}[Y],\ a, b \in \mathbb{R}$

Note also that, if X and Y are independent, that:

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Note further that, if $Y = f(X)$, then:

$$\mathbb{E}[Y] = \sum_{x \in \Omega_X} g(x) p_X(x)$$

3.3 Variance

The *variance* of a random variable is defined as (where $\mu = \mathbb{E}[X]$):

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mu^2)] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\sigma_X^2 + \mu^2 = \mathbb{E}[X^2]$$

To compute the variance, we use one of (3) tactics:

1. If $|\Omega|$ is small, and/or $\mathbb{E}[X]$ is easy to compute/known, then for each $\omega \in \Omega$, compute $(X - \mathbb{E}[X])^2$ at each point, and take the weighted sum (by ω).
2. Numerically manipulate the second equality above.
3. Use one of the common distributions below.

We say that the standard deviation is given as:

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

Note the following:

$$\mathrm{Var}(aX + b) = a^2\, \mathrm{Var}(X)$$

When random variables X and Y are independent, (see §2.5), then (and only then) the following holds:

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

4 Common Discrete Distributions

4.1 Uniform

We say $X \sim \mathrm{Unif}(a,b)$ when (for $a \leq b$) X has an equally likely chance of taking a value in $[a,b]$. (Example: the outcome of a die roll).

$$p_X(k) = \frac{1}{b-a+1},\ k \in [a,b]$$

$$F_X(x) = \frac{\lfloor x \rfloor - a + 1}{b - a + 1}$$

$$\mathbb{E}[X] = \frac{a+b}{2} \quad \mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

4.2 Bernoulli (indicated)

We say $X \sim \mathrm{Ber}(p)$ when X is either 0 or 1 with probability p.

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \geq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\mathbb{E}[X] = p \quad \mathrm{Var}(X) = p(1-p)$$

4.3 Binomial

We say $X \sim \mathrm{Bin}(n,p)$ when X is the sum of Bernoulli trials. An example of this is the *number* of heads in a series of n coin flips.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k},\ k \in [0,n]$$

$$F_X(x) = \sum_{i=0}^{\lfloor x \rfloor} \binom{n}{k} p^i (1-p)^{n-i}$$

$$\mathbb{E}[X] = np \quad \mathrm{Var}(X) = np(1-p)$$

Note that $\mathrm{Bin}(1,p) \triangleq \mathrm{Ber}(p)$. Note further that as $n \to \infty$ and $p \to 0$, that:

$$\lim_{\substack{n \to \infty \\ p \to 0}} \mathrm{Bin}(n,p) = \mathrm{Poi}(\lambda),\ np = \lambda$$

4.4 Geometric

We say that $X \sim \mathrm{Geo}(p)$ when X is the number of Bernoulli trials up to the first successful trials.

$$p_X(k) = (1-p)^{k-1} p,\ k \in \mathbb{N}$$

$$\mathbb{E}[X] = \frac{1}{p} \quad \mathrm{Var}(X) = \frac{1-p}{p^2} \quad F_X(x) = 1 - (1-p)^k$$

4.5 Poisson

We say that $X \sim \mathrm{Poi}(\lambda)$ when X is the number of events during a given time, where λ is the rate of said event.

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!},\ k \in [0, \infty)$$

$$\mathbb{E}[X] = \mathrm{Var}(X) = \lambda$$

Note that if $X_i \sim \mathrm{Poi}(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim \mathrm{Poi}(\lambda_1 + \ldots + \lambda_n)$.