MATH126 D, Taggart, Final Exam

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Contents

Chapter 10

Parametric Equations, Polar Coordinates

10.1 Parametric Curves

To represent a non-function curve C, it is impossible to do so by defining the curve $C = f(x)$ for two values of x that produce different output. Instead, we define the following functions in $\mathbb{R}^3$

$$
\begin{aligned}
x &= f(t) \\
y &= g(t) \\
z &= h(t)
\end{aligned}
$$

Importantly, these classes of functions have two qualities:

Direction The path traced out by a parametric equation in $\mathbb{R}^n$ is not direction-less. The direction can be found by evaluating the function at different values for the parameter, t.

Speed The path traced out by a parametric equation is not traced at linear speed. In other words, the space between each continuous point on the curve is not constant.

10.1.1 Parameter elimination

To write a parametric function in $\mathbb{R}^2$ using the function $y = f(x)$, first find the parameter t in terms of x, and then substitute into the equation for y to determine y as a function of x.

Algebraic substitution The most direct strategy for representing a parametric function as a single equation is by algebraic substitution. Solve for y in terms of t, and x in terms of t, then solve for y in terms of x. This may yield multiple equations, if the output is not strictly a function. Graph both.

xt-, yt-, zt-plane graphs Graph each function in the xt-, yt-, zt-planes, and use the visual intuition that for a given input t, the coordinate output of the function will be the result of the plane graphs at the same point t.

Trig-identities If the function is of the form (e.g., $\sin^2(t) + \cos^2(t)$), then you can use the Pythagorean trigonometric identity, and yield a circle.

10.2 Calculus with Parametric Curves

10.2.1 Differentiation

To find the derivative of a curve defined by:

$$y = f(t)$$
$$x = g(t)$$

note that:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \tag{10.1}$$

therefore:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{10.2}$$

This yields the following properties:

1. The slope of a parametric function can be found without the parameter elimination tactic.
2. The horizontal tangents of a parametric function can be found when $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$.
3. The vertical tangents of a parametric function can be found when $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

10.2.2 Area, integration

The area under a given parametric function as defined above can be found using standard technique. Note that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(t) \iff \mathrm{d}x = f'(t)\,\mathrm{d}t \tag{10.3}$$

Recall also, that:

$$A = \int_a^b y\,\mathrm{d}x = \int_a^b x\,\mathrm{d}y \tag{10.4}$$

therefore:

$$A = \int_a^b g(t)f'(t)\,\mathrm{d}t \tag{10.5}$$

10.2.3 Arc Length

Note that a little piece of the curve traced out by the parametric function can be defined as:

$$\sqrt{\frac{\mathrm{d}x}{\mathrm{d}t}^2 + \frac{\mathrm{d}y}{\mathrm{d}t}^2} \tag{10.6}$$

Therefore, the length of the arc traced out from α to β is given as:

$$\int_\alpha^\beta \sqrt{\frac{\mathrm{d}x}{\mathrm{d}t}^2 + \frac{\mathrm{d}y}{\mathrm{d}t}^2}\,\mathrm{d}t \tag{10.7}$$

10.2.4 Surface Area

To find the surface area of a solid of rotation given by a parametric equation, recall that we can subdivide the solid into a inf-amount of frustums, each of which has an area given by:

$$A = 2\pi rh = 2\pi y \mathrm{d}s \tag{10.8}$$

Therefore, the surface area of a solid of rotation given by the parametric equations above is:

$$A = \int_{\alpha}^{\beta} 2\pi g(t) \sqrt{\frac{\mathrm{d}x}{\mathrm{d}t}^2 + \frac{\mathrm{d}y}{\mathrm{d}t}^2}\, \mathrm{d}t \tag{10.9}$$

Generally, these equations are given in the form:

About x-axis $S = \int 2\pi y \, \mathrm{d}s$

About y-axis $S = \int 2\pi x \, \mathrm{d}s$

10.3 Polar Coordinates

The polar coordinate system is an alternative and convenient way of graphing trigonometric functions. It contains a polar axis, the basis of angle measurements where $\theta = 0$.

To graph a point $P(r, \theta)$, move θ radians counter-clockwise from the pole, and then r units away from $O(0, 0)$.

Given this, we know that the following points are equivalent:

$$P(r, \theta) = P(r, \theta + 2n\pi) = P(-r, \theta + (2n + 1)\pi) \tag{10.10}$$

10.3.1 Converting to Cartesian Coordinates

To convert from polar to Cartesian coordinates, note that:

$$x = r \cos \theta \tag{10.11}$$

and:

$$y = r \sin \theta \tag{10.12}$$

10.3.2 Converting to Polar Coordinates

Given a point in the Cartesian plane $P(x, y)$, note that:

$$r^2 = x^2 + y^2 \rightarrow r = \pm\sqrt{x^2 + y^2} \tag{10.13}$$

by the Pythagorean theorem. Note also that:

$$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{y}{x} \tag{10.14}$$

10.3.3 Polar Equations

Circles

$r(\theta) = a\sin\theta$ Circle with diameter a on the right-side of vertical axis.

$r(\theta) = a\cos\theta$ Circle with diameter a on the top-side of horizontal axis.

$r(\theta) = a\sin$ Circle with radius a placed around the origin.

Limaçons

Limaçons have either one of two forms:

1. $r(\theta) = a \pm b\sin\theta$
2. $r(\theta) = a \pm b\cos\theta$

Where the $\sin\theta$ Limaçons are placed with the information side on:

- If $\sin\theta$ is positive: Above the horizontal axis.
- If $\sin\theta$ is negative: Below the horizontal axis.
- If $\cos\theta$ is positive: Right-side of the vertical axis.
- If $\cos\theta$ is negative: Left-side of the vertical axis.

And contain the following information sides:

- If $\frac{a}{b} < 1$: Complete loop.
- If $\frac{a}{b} = 1$: Heart-shaped loop.
- If $1 < \frac{a}{b} < 2$: Soft loop.
- If $\frac{a}{b} \geq 2$: Flat side.

Rose curves

A rose curve has either one of two forms:

1. $r(\theta) = a \sin(n\theta)$
2. $r(\theta) = a \cos(n\theta)$

If the curve has a $\sin\theta$ term, then it loops *around* the polar axis. Otherwise, if the curve is of the form $\cos\theta$, it loops *over* the polar axis. If n is even, then the rose has $2n$ pedals. Otherwise, if n is odd, then the curve has n pedals.

Lemniscates

A lemniscate has either one of two forms:

1. $r(\theta) = a^2 \sin(2\theta)$
2. $r(\theta) = a^2 \cos(2\theta)$

Where $\sin(2\theta)$-containing functions wrap around the line $\theta = \frac{\pi}{2}$, and $\cos(2\theta)$-containing functions wrap around the polar axis.

10.3.4 Polar Differentiation

To find the derivative of a polar function $r(\theta)$, note that:

$$x = r\cos\theta y = r\sin\theta \tag{10.15}$$

also note that:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \tag{10.16}$$

therefore:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta} r\sin\theta}{\frac{\mathrm{d}}{\mathrm{d}\theta} r\cos\theta} \tag{10.17}$$

by application of The Chain Rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \tag{10.18}$$

Chapter 12

Vectors & the Geometry of Space

12.1 $\mathbb{R}^3$-Space Coordinate Systems

A point $P(x, y, z)$ is represented in $\mathbb{R}^3$ space as:

1. x-units away from the origin along the positive x-axis.
2. y-units away from $(x, 0, 0)$ along the positive y-axis.
3. z-units away from $(x, y, 0)$ along the positive z-axis.

Whereas $\mathbb{R}^2$ space is composed of two coordinate *axes*, $\mathbb{R}^3$ is composed of three coordinate *planes*. The intersection of these planes form 8-octants.

12.1.1 Right-hand rule

Curling your right hand around from the x-axis to the positive extent of the y-axis leaves your right thumb pointing in the direction of the positive extent of the z-axis.

12.1.2 Planar projections

To project $P(x, y, z)$ onto a plane, apply the coordinates $P_a(x, y, z)$ substituting the non-represented variable (i.e., the x coordinate is unused for the yz-plane) to be equal to zero.

As such, projecting $P(x, y, z)$ onto the following planes gives:

xy**-plane** : $P(x, y, 0)$

xz**-plane** : $P(x, 0, z)$

yz**-plane** : $P(0, y, z)$

12.1.3 $\mathbb{R}^3$ formalized

A formal definition of $\mathbb{R}^3$ follows, as the Cartesian product of three sets of real numbers:

$$\mathbb{R}^3 = \left\{(x, y, z)|x, y, z \in \mathbb{R}^3\right\} \tag{12.1}$$

12.1.4 Equations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Equations describe different shapes (curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$) in different dimensions. For instance, the equation $y = c$ defines a horizontal line in $\mathbb{R}^2$, but defines instead a vertical plane in $\mathbb{R}^3$.

12.1.5 Distance in $\mathbb{R}^3$

$$|P_1P_2| = \sqrt{\{x_2 - x_1\}^2 + \{y_2 - y_1\}^2 + \{z_2 - z_1\}^2} \tag{12.2}$$

12.1.6 Spheres in $\mathbb{R}^3$

A sphere is given to be:

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2 \tag{12.3}$$

Where $C(x_1, y_1, z_1)$ is the center of the sphere, and r is the radius of that sphere.

12.2 Vectors

A vector has both a magnitude and a direction. It is commonly represented as an arrow in space. A displacement vector is a special case of the prior definition, where each endpoint of the vector corresponds to a starting and ednting position of a body in space, respectively.

12.2.1 Vector equality

Two vectors are equal when both magnitudes are equal, and both directions are equal. Note: this does *not* compare the starting and ending points of a vector, only distance traveled between them.

12.2.2 Vector summation

To add two vectors together, apply one of the following rules:

Triangle Rule when two vectors can be arranged "tip-to-tail", line them up as so, and draw from the start of one to the end of another. Let this be the vector summation of the two.

Parallelogram Rule when two vectors can be arranged "tip-to-tip", line them up as so, draw a parallelogram containing the two as sides, and then draw across the major diagonal of the parallelogram. Let this be the vector summation of the two.

12.2.3 Scalar multiplication

Let c be a scalar, and $\vec{v}$ be $\langle x_1, y_1, z_1 \rangle$:

$$c\vec{v} = c\langle x_1, y_1, z_1 \rangle = \langle cx_1, cy_1, cz_1 \rangle \tag{12.4}$$

12.2.4 Component of $\vec{v}$

From $O(0, 0, 0)$ to $A(x, y, z)$, let the vector be:

$$\vec{v} = \langle x, y, z \rangle \tag{12.5}$$

From any point $A(x_1, y_1, z_1)$ to any other point $B(x_2, y_2, z_2)$, let the vector between them be defined as:

$$\vec{v} = \vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{12.6}$$

Recall that these two are the same, since vectors only retain their magnitude and direction, and thus can be moved anywhere in $\mathbb{R}^n$.

12.2.5 Magnitude of $\vec{v}$

In $\mathbb{R}^2$:

$$|\vec{v}| = \sqrt{x^2 + y^2} \tag{12.7}$$

and in $\mathbb{R}^3$:

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2} \tag{12.8}$$

12.2.6 Addition and subtraction

Let $\vec{a} = \langle x_1, y_1, z_1 \rangle$, and $\vec{b} = \langle x_2, y_2, z_2 \rangle$. Addition and subtraction are defined as:

$$\vec{a} + \vec{b} = \langle x_2 + x_1, y_2 + y_1, z_2 + z_1 \rangle \tag{12.9}$$

$$\vec{a} - \vec{b} = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \tag{12.10}$$

12.2.7 Scalar multiplication

Let $\vec{a} = \langle x_1, y_1, z_1 \rangle$, and c be a constant. Scalar multiplication is defined over $\mathbb{R}^n$-vectors as follows:

$$c\vec{a} = \langle cx_1, cy_1, cz_1, ... \rangle \tag{12.11}$$

12.2.8 Arithmetic properties of vectors

The following properties arise:

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- $\vec{a} + \vec{0} = \vec{a}$
- $\vec{a} + -\vec{a} = 0$
- $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$

- $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
- $(cd)\vec{a} = c(d\vec{a})$
- $1\vec{a} = \vec{a}$

12.2.9 Standard basis vectors

x-**axis** : $\hat{i} = \langle 1, 0, 0 \rangle$

y-**axis** : $\hat{j} = \langle 0, 1, 0 \rangle$

z-**axis** : $\hat{k} = \langle 0, 0, 1 \rangle$

12.2.10 Arbitrary unit vectors

The unit vector (read: normalization) of vector $\vec{v}$ is another vector $\vec{u}$ in the same direction as $\vec{v}$ and having magnitude 1, as defined:

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} \tag{12.12}$$

12.3 Dot product

Let the dot product of two vectors, $\vec{a} = \langle a_1, a_2, a_3 \rangle$, and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be:

$$\vec{a} \cdot \vec{b} = (a_1 b_1) + (a_2 b_2) + (a_3 b_3) \tag{12.13}$$

12.3.1 Arithmetic properties

The following properties follow:

$$\begin{aligned}
\vec{a} \cdot \vec{a} &= |\vec{a}|^2 \\
&= a_1 a_1 + a_2 a_2 + a_3 a_3 \\
&= a_1^2 + a_2^2 + a_3^2 \\
&= \sqrt{a_1^2 + a_2^2 + a_3^2}^2
\end{aligned}$$

$$
\begin{aligned}
\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
&= b_1 a_1 + b_2 a_2 + b_3 a_3
\end{aligned}
$$

$$
\begin{aligned}
\vec{a} \cdot (\vec{b} + \vec{c}) &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \\
&= a_1(a_2 + a_3) + b_1(b_2 + b_3) + c_1(c_2 + c_3) \\
&= a_1 a_2 + a_1 a_3 + b_1 b_2 + b_1 b_3 + c_1 c_2 + c_1 c_3 \\
&= (a_1 a_2 + b_1 b_2 + c_1 c_2) + (a_1 a_3 + b_1 b_3 + c_1 c_3)
\end{aligned}
$$

$$
\begin{aligned}
(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) &= \vec{a} \cdot (c\vec{b}) \\
&= ca_1 b_1 + ca_2 b_2 + ca_3 b_3 \\
&= c(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= a_1(cb_1) + a_2(cb_2) + a_3(cb_3)
\end{aligned}
$$

$$
\begin{aligned}
\vec{a} \cdot \vec{0} &= 0a_1 + 0a_2 + 0a_3 \\
&= 0
\end{aligned}
$$

12.3.2 The angle between two vectors

The dot product between vectors $\vec{a}$ and $\vec{b}$ can also be defined in terms of The Law of Cosines as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \tag{12.14}$$

A proof follows between the vectors $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{AC}$.

$$\begin{aligned}
|\vec{BC}|^2 &= |\vec{AB}|^2 + |\vec{AC}|^2 - 2|\vec{AB}||\vec{AC}|\cos\theta \\
|\vec{u} - \vec{v}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta \\
(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot \vec{u} - 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v} \\
\vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}|\cos\theta
\end{aligned}$$

12.3.3 Orthogonality

$\vec{v} \cdot \vec{u} = 0$ When two vectors $\vec{v}$ and $\vec{u}$ are orthogonal, the angle θ between them must be $\theta = \pi$. Therefore, the dot product of the two must equal 0.

$\vec{v} \cdot \vec{u} = |\vec{a}||\vec{b}|$ When two vectors $\vec{v}$ and $\vec{u}$ are in the same direction, the angle θ between them must equal 0. Therefore, $\cos\theta = \cos 0 = 1$, and the dot product must be equal to the scalar product of their two magnitudes.

$\vec{v} \cdot \vec{u} = -|\vec{a}||\vec{b}|$ When two vectors $\vec{v}$ and $\vec{u}$ are in the opposite direction, the angle θ between them must equal π. Therefore, $\cos\theta = \cos 0 = -1$, and the dot product must be equal to the opposite of the scalar product of their two magnitudes.

12.3.4 Direction cosines

For a given vector $\vec{v}$, the angles α β, and γ are angles formed with the positive extents of the x-, y- and z-axes, respectively:

$$\cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}||\hat{i}} = \frac{a_1}{|\vec{a}|} \tag{12.15}$$

$$\cos\beta = \frac{\vec{a} \cdot \hat{j}}{|\vec{a}||\hat{j}} = \frac{a_2}{|\vec{a}|} \tag{12.16}$$

$$\cos\gamma = \frac{\vec{a} \cdot \hat{k}}{|\vec{a}||\hat{k}} = \frac{a_3}{|\vec{a}|} \tag{12.17}$$

or, conveniently:

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= |\vec{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

12.3.5 Projections

The projection of $\vec{b}$ onto $\vec{a}$ is the component of $\vec{b}$ along $\vec{a}$. Geometrically, this is taken to be:

1. Taking the terminal of $\vec{b}$ down to the perpendicular of $\vec{a}$.
2. Drawing a vector from the origin of $\vec{a}$ to the intersection of s.

$$\text{proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\frac{1}{|\vec{a}|}\vec{a}\right) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} \tag{12.18}$$

Likewise, the scalar component of this projection is given as:

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \tag{12.19}$$

12.4 Cross product

To find a vector perpendicular to both $\vec{u}$ and $\vec{v}$, take the cross product:

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}\end{aligned} \tag{12.20}$$

12.4.1 Orthogonality

The cross product of $\vec{u}$ and $\vec{v}$ ($\vec{u} \times \vec{v}$) is given to be orthogonal to both $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0 \tag{12.21}$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = 0 \tag{12.22}$$

12.4.2 Magnitude

The magnitude of $\vec{u} \times \vec{v}$ is given to be:

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta \tag{12.23}$$

where θ is the angle in radians between $\vec{u}$ and $\vec{v}$.

Parallel cross products

Therefore, the cross product of two parallel vectors *must* be 0, since $\theta = 0$ (or $\theta = \pi$, depending on the domain restrictions), and $\sin\theta = \sin 0 = 0$.

12.4.3 Applications

Parallelogram area

The area of a parallelogram with skewed edges $\vec{u}$ and $\vec{v}$ is given to be $\vec{u} \times \vec{v}$. If $\vec{c}$ is the perpendicular of $\vec{v}$ to the height of $\vec{u}$ as:

$$\vec{c} = \vec{u} \sin\theta \tag{12.24}$$

then the area of the parallelogram is

$$\begin{aligned} |\vec{v}||\vec{c}| &= |\vec{v}| \left(|\vec{u}| \sin\theta\right) \\ &= |\vec{u} \times \vec{v}| \end{aligned}$$

Area of a triangle

To find the area of a triangle given three points (or two vectors), apply the same process as above to find the area of a parallelogram, and divide by two (since the area of a triangle divides the area of a parallelogram into two).

12.4.4 Properties of the cross product

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $c\vec{a} \times \vec{b} = c\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(c\vec{b}\right)$
- $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\left(\vec{a} + \vec{b}\right) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$
- $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (\vec{a} \cdot \vec{c})\,\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$

12.4.5 Triple product

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 = b_3c_1) + a_3(b_1c_2 - b_2c_1) \tag{12.25}$$

Volume of a parallelepiped

Consider the volume of space bounded by three vectors: $\vec{a}$, $\vec{b}$, and $\vec{c}$. Let $\vec{a}$ and $\vec{b}$ define the parallelepiped's base parallelogram, and let $\vec{c}$ give the slanted height of the volume.

The volume can be calculated as such:

$$h = |\vec{a}||\cos\theta| \tag{12.26}$$

$$A = |\vec{c} \times \vec{b}| \tag{12.27}$$

$$V = hA = |\vec{b} \times \vec{c}||\vec{a}||\cos\theta| = |\vec{a} \cdot (\vec{b} \times \vec{c})| \tag{12.28}$$

12.5 Equations of Lines & Planes in Space

12.5.1 Lines in Space

To find the equation for **L** given by some initial point $P(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$, take the following:

1. Find the vector $\vec{r_0}$ from the origin to the point $P_0(x_0, y_0, z_0)$.
2. The get another vector from the origin to $P(x, y, z)$, add some scalar multiple of $\vec{v}$ to $\vec{r_0}$.

Therefore:

$$\vec{r} = \vec{r_0} + t\vec{v} \tag{12.29}$$

alternatively:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle \tag{12.30}$$

or:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{12.31}$$

Line segments in space

A line segment from $\vec{r_0}$ to $\vec{r_1}$ is given as **L** from $t \in [0, 1]$, or:

$$\begin{aligned}\vec{r} &= \vec{r_0} + t\vec{v} \\ &= \vec{r_0} + t(\vec{r_1} - \vec{r_0}) \\ &= \vec{r_0}(1 - t) + t\vec{r_1}\end{aligned}$$

12.6 Planes in Space

Let $P(x, y, z)$ be an arbitrary point on a plane **P**. Let $\vec{n}$ be the normal vector of the plane. Let $P_0(x_0, y_0, z_0)$ be a known point on the plane:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0 \tag{12.32}$$

Letting the normal vector $\vec{n} = \langle a, b, c \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{12.33}$$

The distance between two planes with equations $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$:

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^3}} \tag{12.34}$$

12.6.1 Applications

Given $P_0(x_0, y_0, z_0)$ and $\vec{n}$ Find the plane through the given point and normal to the vector $\vec{n}$ using the above equation.

Given three points Find a vector normal to two vectors given from the three points by taking the cross product as $\vec{v} = \overrightarrow{P_0 P_1}$, $\vec{u} = \overrightarrow{P_0 P_2}$, and $\vec{v} \times \vec{u}$. Use any three of the points and the cross product in the above.

Angle between two planes Take the angle between the two normal vectors $\vec{n_1}$ and $\vec{n_2}$ by using the $\cos\theta$ definition of the dot-product:

$$\theta = \arccos\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right) \tag{12.35}$$

Distance to a point Take the scalar projection of the vector from a point on the plane to the given point (let this be $\vec{b}$) onto the normal vector of the plane (let this be $\vec{n}$), as:

$$\text{comp}_{\vec{n}}\,\vec{b} = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \tag{12.36}$$

Distance between two planes Take any point on the second plane, then see above.

Distance between two skew lines Take each line to be in a plane that is parallel to each other, then see above.

12.6.2 Problem sets (§12.5-2)

1. Given a point $P(x, y, z)$ and a plane with a known normal vector, use the standard form of the planar equation to construct a new plane.

2. Given three points, construct two vectors ($\vec{u}$ and $\vec{v}$) and take the cross product to be the normal vector ($\vec{n} = \vec{u} \times \vec{v}$). Use any of the three points to construct a planar equation.

3. Solve the planar equation at $P_1(x, 0, 0)$, $P_2(0, y, 0)$, and $P_3(0, 0, z)$ to match with the provided graph.

4. Find the cosine of the angle between two planes by taking the dot product of each plane's normal vectors $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

5. To find the plane where all points are equidistant from two other points, pick an arbitrary point $P(x, y, z)$ and set the distance formula equal to the two provided points. Clean up the resulting relationship to the standard planar equation.

6. Given desired intercepts, construct three points (one for each coordinate axis) and determine a plane given the three points.

7. Use the distance between two planes (see: above) to find two different values for d_2.

12.6.3 Problem sets (§12.5-3)

1. To find the equation for a line perpendicular to a plane, use the given point and the normal vector for that plane, given in the standard planar equation.

2. To find the equation of a plane that contains a line and a given point, solve the point at $t = 0$ and $t = 1$ to find new points, construct then a plane using all three points (two vectors, cross product, etc).

3. To find when a line intersects a plane, solve the equation of the plane using the parametric form of the line.

4. To find where a line composed of two points intersects a plane, find the line using the initial point and a displacement vector, then see above.

5. To find the line of intersection given a plane, solve the system of equations for two planes letting one variable be equal to a constant (this will always hold true for intersections, but never for parallel planes). Solve the system for parametric equations. To find the angle between the two, use the cos-definition of the dot product among $\vec{n_1}$ and $\vec{n_2}$.

6. To find the plane containing a point and the intersection of two other planes, solve for two points on the intersection of the two planes, construct two vectors from three lines, take the cross product for the normal vector, and then use the standard equation of the plane.

7. To find the intersection of two parametric lines in $r(t)$ and $r(s)$, set the two's components equal to each other's respective components, and find t in terms of s. Then solve for values of t and s that satisfy the six equations.

8. To find the distance between a point and a plane, take the scalar projection of a displacement vector to that point and the normal vector on the plane $\text{comp}_{\vec{n}}\,\vec{b}$.

12.7 Cylinders & Quadratic Surfaces

Surfaces are complex surfaces in $\mathbb{R}^3$-space that have traces of curves in $\mathbb{R}^2$ parallel to each of the coordinate planes.

12.7.1 Cylinders

A cylinder is the surface that consists of all lines (rulings) that are parallel to a given line and pass through a plane curve.

Cylinders can be traditional-looking (i.e., $x^2 + y^2 = 1$) or can be of other shapes, for instance $z = x^2$ which is a successive set of parabolic traces parallel to the y-axis.

Note that, in $\mathbb{R}^3$, $x^2 + y^2 = 1$ is the equation $x^2 + y^2 = 1$ and the fact $z = k \in \mathbb{R}$. To produce a tracing of the $\mathbb{R}^2$ representation in $\mathbb{R}^3$, use $x^2 + y^2 = 1$, $z = 0$.

12.7.2 Quadratic Surfaces

Quadratic surfaces are second-degree polynomials in three variables:

$$Ax^2 + By +^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \quad (12.37)$$

which can be generalized to:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad (12.38)$$

or:

$$Ax^2 + By^2 + Iz = 0 \quad (12.39)$$

To sketch a quadratic surface:

1. Fix one of the free variables (for instance, $x = k$).
2. Examine the resulting $\mathbb{R}^2$ trace, and consider what occurs as the fixed variable moves to larger and smaller values.
3. Repeat this process for remaining un-fixed variables.
4. Trace the resulting shape.

Known surfaces

Ellipsoid Has an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (12.40)$$

Where all traces are ellipses. If $a = b = c$, then the ellipsoid is a sphere.

Elliptic Paraboloid Has an equation of the form:

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (12.41)$$

Where horizontal traces are ellipses and vertical traces are parabolas. The variable of degree 1 indicates the axis of the paraboloid.

Hyperbolic Paraboloid Has an equation of the form:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \tag{12.42}$$

Where horizontal traces are hyperbolas and vertical traces are parabolas.

Cone Has an equation of the form:

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \tag{12.43}$$

Where horizontal traces are ellipses, and vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$, but are pairs of lines if $k = 0$.

1-Hyperboloid Has an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{12.44}$$

Where horizontal traces are ellipses, and vertical traces are hyperbolas. The axis of symmetry corresponds to which variable has a negative coefficient.

2-Hyperboloid Has an equation of the form:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{12.45}$$

Where horizontal traces are ellipses in $z = k$ if $k > c$ or $z < -c$. Vertical traces are hyperbolas.

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

A vector-valued function (or, vector function) is a parametric function whose domain is a set of real numbers and whose range is a set of vectors. This can be defined parametrically as:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \tag{13.1}$$

The limit of this function is taken as:

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle \tag{13.2}$$

13.1.1 Continuity

A vector function $\vec{r}$ is continous at a if:

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a) \tag{13.3}$$

Suppose that $\vec{r}(t)$ is continuous over all points in I. Therefore, a curve C (a space curve) is defined to be all points (x, y, z) that satisfy those parametric equations above.

13.2 Derivatives and integrals of vector functions

13.3 Derivatives

The derivative of a vector-valued function is given for:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \tag{13.4}$$

as:

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t) \rangle \tag{13.5}$$

And follows the following differentiation rules given that c is a scalar constant, f is a real-valued function, and u and v are vector-valued functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{v'}(t) + \vec{u'}(t) \tag{13.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[c\vec{u}(t)\right] = c\vec{u'}(t) \tag{13.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\vec{v}(t)\right] = f'(t)\vec{v}(t) + f(t)\vec{v'}(t) \tag{13.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u'}(t) \cdot \vec{v}(t) + \vec{v'}(t) \cdot \vec{u}(t) \tag{13.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u'}(t) \times \vec{v}(t) + \vec{v'}(t) \times \vec{u}(t) \tag{13.10}$$

13.3.1 Integrals

The integral of a vector-valued function $r(t)$ as given above, is defined as:

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \langle \int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t, \int_a^b h(t)\,\mathrm{d}t \rangle \tag{13.11}$$

13.4 Arc Length & Curvature

The length of a space curve in $\mathbb{R}^3$ given as $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined to be:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, \mathrm{d}t \tag{13.12}$$

Notice, however, that:

$$\vec{r}'(t) = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \tag{13.13}$$

therefore:

$$L = \int_a^b \vec{r'}(t)\, \mathrm{d}t \tag{13.14}$$

13.4.1 Re-parameterization

Let the arc-length function from a to t be defined as:

$$s(t) = \int_a^t \vec{r'}(u)\, \mathrm{d}u \tag{13.15}$$

Given this, it is often useful to re-parameterize the function $\vec{r}(t)$ in terms of arc-length, s, not time elapsed, t. To do so, note:

$$t = t(s) \to r(t) = r(t(s)) \to r(s) \tag{13.16}$$

13.4.2 Curvature

Let the unit tangent vector for a curve C be given as:

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} \tag{13.17}$$

The curvature, κ of a given space curve C is given as:

$$\kappa = \left| \frac{\mathrm{d}T}{\mathrm{d}s} \right| \tag{13.18}$$

or defined by The Chain Rule, as:

$$\kappa = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3} \tag{13.19}$$

Let the unit normal vector be defined as:

$$\vec{N} = \frac{1}{\vec{T'}(t)} \vec{T'}(t) \tag{13.20}$$

The binormal vector, orthogonal to both the unit tangent and unit normal vectors is given as:

$$\vec{B} = \vec{T} \times \vec{N} \tag{13.21}$$

Normal Plane

This plane contains all lines orthogonal to the space curve C at $P(x, y, z)$. It is thought to be the "worst" approximation to the curve at that point. It contains the binormal and normal vectors, thus:

$$T_1(x - x_1) + T_2(y - y_1) + T_3(z - z_1) = 0 \tag{13.22}$$

Osculating Plane

This plane is closest to the space curve C at $P(x, y, z)$. It is thought to be the "best" approximation to the curve at that point. It contains the tangent and normal vectors, and thus:

$$B_1(x - x_1) + B_2(y - y_1) + B_3(z - z_1) = 0 \tag{13.23}$$

Osculating Circle

This circle is defined for $\mathbb{R}^2$ curves as the closest circle to a function $x = f(t)$, $y = g(t)$ for which the circle contains the same point of intersection, tangency, and curvature.

$$(x(t) - \frac{1}{\kappa(t)})^2 + (y(t) - \frac{1}{\kappa(t)})^2 = \frac{1}{\kappa(t)^2} \tag{13.24}$$

13.5 Motion in Space: Velocity & Acceleration

Recall that, for a position function:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \tag{13.25}$$

Its velocity function is defined as:

$$\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t) = \langle x'(t), y'(t), z'(t) \rangle \tag{13.26}$$

Note: the magnitude of velocity v is defined as:

$$v = |\vec{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \tag{13.27}$$

And its acceleration function is defined as:

$$\vec{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) = \langle x''(t), y''(t), z''(t) \rangle \tag{13.28}$$

Similarly:

$$\vec{v}(t) = \int \vec{a}(t)\,\mathrm{d}t \tag{13.29}$$

as well:

$$\vec{r}(t) = \int \vec{v}(t)\,\mathrm{d}t \tag{13.30}$$

Recall, however, that integrating produces a constant of integration in each dimension (C_1, C_2, etc.) and must be determined to be a linear scalar quantity at each step.

13.5.1 Newton's Second Law of Motion

Recall that Newton's Second Law of Motion is stated as:

$$\vec{F} = m\vec{a} \tag{13.31}$$

which indicates that $\vec{a}$, m, or $\vec{F}$ can be solved for when values for the other two are known.

13.6 Tangential and Normal Components

It is useful to express the components of acceleration ($\vec{a}$) in terms of a scalar quantity along the tangent vector ($\vec{T}$) and the normal vector ($\vec{N}$):

$$a_T = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{|\vec{r'}(t)|} \tag{13.32}$$

$$a_N = \frac{\vec{r'}(t) \times \vec{r''}(t)}{|\vec{r'}(t)|} \tag{13.33}$$

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

A function f fo two variables assigns each (x, y) pair to an output in the range of f, such that:

$$\{f(x, y)|(x, y) \in D\} \tag{14.1}$$

additionally:

$$\{(x, y) \subseteq \mathbb{R}^2, f(x, y) \subseteq \mathbb{R}\} \tag{14.2}$$

14.1.1 Linear Functions of Three Variables

A linear functions is defined as:

$$Ax + By + C = z \tag{14.3}$$

14.1.2 Level Curves

A level curve for $f(x, y)$ at k (read: the k-th level curve for f) is given as:

$$\{(x, y) \subseteq \mathbb{R}^2|f(x, y) = k\} \tag{14.4}$$

or, alternatively:

$$f(x, y) = k \tag{14.5}$$

14.1.3 Higher Dimensions

Functions can take place in arbitrarily-high dimensions. For instance, the function f:

$$f(x_1, x_2, \ldots, x_n) \tag{14.6}$$

maps inputs in the domain set to the n-tuple:

$$(x_1, x_2, \ldots, x_n, z) \tag{14.7}$$

such that:

$$\{x_1, x_2, \ldots, x_n \subseteq \mathbb{R}^n | f(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1}\} \tag{14.8}$$

Or, in other words, a function f increases dimensionality from its input space of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ in its output space.

14.2 Partial Derivatives

Derivates for higher dimension functions are taken one variable at a time. The algorithm is as follows:

1. To differentiate a multivariate function, take the remaining free variables and for each:
2. Differentiate the function with respect to that variable, leaving all other expressions as constants.
3. Repeat until there are no more remaining free variables.

There are several ways to notate this operation:

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \tag{14.9}$$

14.3 Tangent Planes and Linear Approximations

Here, we present an analogue of approximate curves in $\mathbb{R}^2$ with a linear approximation to approximating surfaces in $\mathbb{R}^3$ with a tangent plane that behaves similarly to the function $z = f(x, y)$ "near" point $P_0(x_0, y_0, f(x_0, y_0))$.

14.3.1 Tangent Planes

Note that the plane will contain a vector normal to the surface at the point ($\vec{n} = \langle n_1, n_2, n_3 \rangle$), and the point itself ($P_0(x_0, y_0, f(x_0, y_0))$), of the form:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \tag{14.10}$$

Let $\vec{r}_x(t)$ be the path around the curve along the x-axis, as follows:

$$\vec{r}_x(t) = \langle t, y_0, f(t, y_0) \rangle \tag{14.11}$$

Further, let $\vec{r}_y(t)$ be the path around the curve along the x-axis, as follows:

$$\vec{r}_y(t) = \langle x_0, t, f(x_0, t) \rangle \tag{14.12}$$

Note that the first derivatives of these functions will yield a set of tangent vectors along each coordinate axis for the surface in question:

$$\vec{r}'_x(t) = \langle 1, 0, f_x(t, y_0) \rangle \tag{14.13}$$

$$\vec{r}'_y(t) = \langle 0, 1, f_y(x_0, t) \rangle \tag{14.14}$$

Further note that their cross-product will yield a vector tangent to the surface at the point $P_0(x_0, y_0, f(x_0, y_0))$:

$$\vec{n} = \vec{r}'_x \times \vec{r}'_y = \langle f_x(t, y_0), f_y(x_0, t), 1 \rangle \tag{14.15}$$

Therefore, the plane of tangency for the surface $z = f(x, y)$ in $\mathbb{R}^3$ is given as:

$$f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) - (z - z_0) = 0 \tag{14.16}$$

14.3.2 Linear Approximations

Let the tangent plane be used to approximate the surface $z = f(x, y)$ in $\mathbb{R}^3$.

Expand the equation above:

$$\begin{aligned} f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) - (z - z_0) = 0 \\ f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) + z_0 = z \end{aligned}$$

Denote z to be the linear approximation ($L(x, y)$) as follows:

$$L(x, y) = z_0 + f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) \tag{14.17}$$

where:

$$L(x, y) \simeq f(x, y) \tag{14.18}$$

and is especially well-behaved around input points near $P(x_0, y_0)$.

14.3.3 Differentiability

Thm. If the partial derivatives f_x and f_y exist near (a, b) and are continuous at a, and b, then f is differentiable at (a, b).

14.3.4 Total differentials

Let $\mathrm{d}z$ be the approximated change in the function $f(x, y)$ along a change $\Delta x = \mathrm{d}x$, and $\Delta y = \mathrm{d}y$

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y \tag{14.19}$$

Or, for equations represented in $\mathbb{R}^n$:

$$\mathrm{d}z = \sum_{i=0}^{n} \frac{\partial z}{\partial d_i}\,\mathrm{d}d_i \tag{14.20}$$

14.4 Optimization

Let the critical points of $z = f(x, y)$ be defined as the places where either f_x and f_y are concave down, concave up, or one of each.

14.4.1 Critical points of f

Let the set of all critical points be denoted:

$$\{(x_0, y_0) : f_x(x_0, y_0) = f_y(x_0, y_0) = 0\} \tag{14.21}$$

or where f_x, f_y evaluated at (x_0, y_0) is undefined.

14.4.2 2nd derivative test

For each critical point, let its classification be defined as:

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} \tag{14.22}$$

$D(x_0, y_0) > 0$ $f(x, y)$ has a local maxima or minima at (x_0, y_0) if f_{xx} is negative, or positive, respectively.

$D(x_0, y_0) < 0$ $f(x, y)$ has a saddle point at (x_0, y_0).S

$D(x_0, y_0) = 0$ Second derivative test has failed.

14.4.3 Extreme Value Theorem

If $f(x, y)$ is defined on a closed, bounded set $D \subseteq \mathbb{R}^2$, then f attains a maximum value at $f(x_1, y_1)$ at $(x_1, y_1) \in D$, and a minimum value at $f(x_2, y_2)$ at $(x_2, y_2) \in D$.

1. Find all critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. Let the largest value of the union of the above be the absolute (global) maximum of the function f, and the smallest be its minimum.

Chapter 15

Multiple Integrals

15.1 Double Integrals Over Rectangles

Recall the definition of an integral over a range of points in $\mathbb{R}^2$:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{15.1}$$

In $\mathbb{R}^3$, the definition is as follows:

Suppose you have a function $z = f(x, y)$ in $\mathbb{R}^3$, and a closed rectangle in the domain of that function:

$$R = \{(x, y) \in \mathbb{R}^2 | x \in [a, b] \wedge y \in [c, d]\} \tag{15.2}$$

Further suppose that you have a surface S defined in the area above or below that region R:

$$S = \{(x, y, z) \in \mathbb{R}^3 | z \in [0, f(x, y)] \wedge (x, y) \in \mathbb{R}^2\} \tag{15.3}$$

Define the i-th and j-th rectangular subdivision of R to be:

$$R_{ij} = \{(x, y) | x \in [x_{i-1}, x_i] \wedge y \in [y_{j-1}, y_j]\} \tag{15.4}$$

such that:

$$\Delta A = \Delta x \Delta y \tag{15.5}$$

Further denote a sample within the i-th and j-th region of R to be: R_{ij}^*

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
$$V = \iint\limits_R f(x, y)\,\mathrm{d}A$$

15.1.1 The Midpoint Rule

Similar to how the midpoints of Riemann sums could be used to approximate the area under a curve for functions in $\mathbb{R}^2$, the midpoint of the i-th and $j-th$ sub-rectangle (denoted $(\bar{x}_i, \bar{y}_i)$) can be used to approximate the volume enclosed in that rectangle:

$$\iint\limits_R f(x, y)\,\mathrm{d}A \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A \tag{15.6}$$

15.1.2 The Average Value Theorem

The average value for a surface in $R \subset \mathbb{R}^2$ can be calculated as the volume of the double-integral divided by the area (denoted as the function $A(R)$) of the area R:

$$f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x, y)\,\mathrm{d}A \tag{15.7}$$

15.2 Iterated Integrals

When presented with a double integral, the contained integrals may be *partially integrated* until the integral is complete:

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \int_a^b \left[\int_c^d f(x, y)\,\mathrm{d}y\right] \tag{15.8}$$

Where the integral taken with respect to $\mathrm{d}y$ is evaluated as if any terms of x were constant. The result of this (with the unevaluated "constant" x-terms) is then integrated with respect to $\mathrm{d}x$.

15.2.1 Fubini's Theorem

Presented without proof, Fubini's Theorem states that the integral of multiple dimensions can be taken in any order, similar to Clairaut's Theorem states that functions of higher dimensions can be differentiated in any order.

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

15.3 Double Integrals over General Regions

15.3.1 Type I Regions

To take the integral of $z = f(x,y)$ in a region D bounded by two continuous functions of x, say: $g_1(x)$, $g_2(x)$, apply the following:

Let:

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in \mathrm{dom}(f) \\ 0 \end{cases} \tag{15.9}$$

therefore:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{15.10}$$

15.3.2 Type II Regions

To take the integral of $z = f(x,y)$ in a region D bounded by two continuous functions of y, say: $g_1(y)$, $g_2(y)$, apply the following:

$$\iint\limits_D f(x,y)\,\mathrm{d}A = \iint\limits_R F(x,y)\,\mathrm{d}A = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{15.11}$$

15.3.3 Properties of Double Integrals

Double integrals are associative over summations of inner functions $f_i(x, y)$:

$$\iint\limits_D [f(x,y) + g(x,y)]\,\mathrm{d}A = \iint\limits_D f(x,y)\,\mathrm{d}A + \iint\limits_D g(x,y)\,\mathrm{d}A \tag{15.12}$$

Double integrals are associative over constants c in the inner functions:

$$\iint\limits_D cf(x,y)\,\mathrm{d}A = c\iint\limits_D f(x,y)\,\mathrm{d}A \tag{15.13}$$

Double integrals are commutative over sub-divisions of the region of integration D. Let $D = D_1 \cup D_2$:

$$\iint\limits_D f(x,y)\,\mathrm{d}A = \iint\limits_{D_1} f(x,y)\,\mathrm{d}A + \iint\limits_{D_2} f(x,y)\,\mathrm{d}A \tag{15.14}$$

Double integrals also hold the following property:

$$\iint\limits_D c\,\mathrm{d}A = cA(D) \tag{15.15}$$

where $A(D)$ is the area of the region D.

Finally, if $m \leq f(x,y) \leq M \forall (x,y) \in D$:

$$mA(D) \leq \iint f(x,y)\,\mathrm{d}A \leq MA(D) \tag{15.16}$$

To determine m and M, use methods from the previous chapter to optimize the function $z = f(x, y)$ over the given domain, D.

Double integrals can be used to find the average value (denoted for $z = f(x,y)$ as f_{avg}), as:

$$f_{avg} = \frac{1}{A(D)} \iint\limits_D f(x,y)\,\mathrm{d}A \tag{15.17}$$

where $A(D)$ is the area given by D, which could be a rectangle, triangle, polar region, or Calculus II function that must itself be integrated.

15.3.4 Order of Integration

It is sometimes useful to reverse the order of integration from $\mathrm{d}x\,\mathrm{d}y$ to $\mathrm{d}y\,\mathrm{d}x$. In order to do so, consider the original region of integration. Rewrite functions of x as functions of y, and vice-versa. Rewrite the integral with the functional limits at the innermost position, and reverse the order of differentials. Integrate.

15.4 Double Integrals in Polar Coordinates

In certain cases, the above "Type I" and "Type II" integrals are not ideal tools to integrate complex general regions. Polar coordinates can be a fit for this instead:

15.4.1 Polar Rectangles

A polar rectangle is a 2-dimensional donut over any arbitrary angle region θ. For instance:

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\} \tag{15.18}$$

15.4.2 Converting Cartesian Functions

Recall the following:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Therefore, the polar conversion of $z = f(x, y)$ to $z = f(r, \theta)$ is given as follows:

$$f(x, y) = f(r\cos\theta, r\sin\theta) \tag{15.19}$$

When f is continuous over a polar rectangle $0 \leq a \leq r \leq b$, and $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$:

$$\iint\limits_D f(x, y)\,\mathrm{d}A = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta \tag{15.20}$$

15.5 Applications of Double Integrals

15.5.1 Density

Recall that the "instantaneous" density for a surface in $\mathbb{R}^3$ is denoted as:

$$\rho(x, y) \tag{15.21}$$

Where the quantity is mass per unit-volume.

Therefore, to calculate the mass m of a surface in $\mathbb{R}^3$ over the sub-domain D, apply:

$$m_D = \iint\limits_D \rho(x, y)\,\mathrm{d}A \tag{15.22}$$

15.5.2 Moments

Recall that a "moment" is defined as the weighted sum of all particles in a surface along a directed axis.

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D y\rho(x, y)\,\mathrm{d}A \tag{15.23}$$

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D x\rho(x, y)\,\mathrm{d}A \tag{15.24}$$

Therefore, the center of mass $(\bar{x}, \bar{y})$ is given to be:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{1}{m}\iint_D x\rho(x,y)\,\mathrm{d}A, \frac{1}{m}\iint_D y\rho(x,y)\,\mathrm{d}A\right) \quad (15.25)$$

Chapter 16

Taylor Polynomials and Taylor Series

16.1 Tangent Line Error Bound

When functions in two dimensions ($y = f(x)$) are too complicated to reason about using traditional calculus, they are reduced by Taylor Approximations.

16.1.1 Tangent Line Approximation ($T_1(x)$)

The first Taylor Approximation of $f(x)$, $T_1(x)$ (centered at $x = b$) is given as:

$$T_1(x) = f(b) + f'(b)(x - b). \tag{16.1}$$

16.1.2 Tangent Line Error Bound

The Tangent Line Approximation ($T_1(x)$) is either exact for linear functions, or only locally well-behaved for more complex functions. To determine the *error* between the approximation, $T_1(x)$, and the function $f(x)$, note that an error bound can be determined as:

$$|f(x) - T_1(x)| \leq \frac{f''(c)}{2}|x - b|^2. \tag{16.2}$$

Where c is a constant such that $f''(c)$ yields the maximum of $f''(x)$ on the interval from $[x, b]$.

To bound the error on a given interval $I = [c - a, c + a]$, note that the error ($|f(x) - T_1(x)|$) is itself a function of x. Maximize $\frac{f''(c)}{2}|x - b|^2$ using methods of traditional Calculus.

16.2 Quadratic Approximation

For more complicated functions, a tangent-line approximation is not accurate enough.

16.2.1 Quadratic Approximation ($T_2(x)$)

For these, a quadratic approximation is introduced, and is defined as $T_2(x)$ for a function $f(x)$ centered at $x = b$.

$$T_2(x) = f(b) + f'(b)(x - b) + \frac{1}{2}f''(b)(x - b)^2. \tag{16.3}$$

16.2.2 Quadratic Approximation Error Bound

We define the *error* to be the difference between $f(x)$ and $T_2(x)$ as:

$$|f(x) - T_2(x)| \leq \frac{f'''(c)}{6}|x - b|^3. \tag{16.4}$$

16.2.3 Properties of the Quadratic Approximation

Note that:

$$T_2(b) = f(b), T_2'(b) = f'(b), T_2''(b) = f''(b) \tag{16.5}$$

16.3 Higher-Order Approximations and Taylor's Inequality

For even more complex functions, a higher-order approximation proves more accurate than $T_1(x)$ or $T_2(x)$.

16.3.1 Higher-Order Approximations

Defined below is the n-th Taylor approximation of $f(x)$ centered at $x = b$:

$$T_n(x) = \sum_{i=0}^{n} \frac{1}{n!} f^{(n)}(b)(x-b)^n. \tag{16.6}$$

16.3.2 Higher-Order Approximation Error Bound

$$|f(x) - T_n(x)| = \frac{f^{(n+1)}(c)}{(n+1)!}|x-b|^{n+1} \tag{16.7}$$

16.4 Taylor Series

We define the Taylor Series for a function $f(x)$ (centered at $x = b$) to be the following:

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(b)(x-b)^n = \lim_{m\to\infty} T_m(x) = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{1}{n!} f^{(n)}(b)(x-b)^n \tag{16.8}$$

Provided that the limit $\lim_{m\to\infty} T_m(x)$ exists.

We say that the Taylor Series for f converges if the limit exists and is finite. We say that the Taylor Series for f diverges if the limit does not exist, or is not finite.

16.4.1 Common Taylor Series

16.5 Operations with Taylor Series

Often times, functions f are difficult or impossible to find a good approximation of using the above Taylor Series. However, if the function f is a composition of other functions (f_1, f_2, and f_n, of which the Taylor Series is known), we can use standard arithmetic operators to Taylor Series.

	Series	Examples
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + ...x^n$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n}$
$\cos(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{n^{2n}}{(2n)!}$	$1 - x + \frac{x^2}{2} + ... + \frac{x^n}{n}$
$\sin(x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{2n-1}}{(2n-1)!}$	$x - \frac{x^3}{3} + ... + \frac{x^{2n-1}}{(2n-1)!}$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^n}{n}$	$x - \frac{x^2}{x} + \frac{x^3}{3} + ... + \frac{x^n}{n}$

16.5.1 Addition, Subtraction, Multiplication

The n-th Taylor Polynomial for the sum of two functions f and g is the sum of their n-th Taylor Polynomials.

Addition

$$h(x) = f(x) + g(x)$$
$$T(h)_n(x) = T(f)_n(x) + T(g)_n(x)$$

Subtraction

$$h(x) = f(x) - g(x)$$
$$T(h)_n(x) = T(f)_n(x) - T(g)_n(x)$$

Multiplication

$$h(x) = f(x) * cg(x)$$
$$T(h)_n(x) = T(f)_n(x) * c * T(g)_n(x)$$

16.5.2 Differentiation, Integration

Taylor Series are also differentiable and integrable.