MATH126 D, Taggart, Exam II

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Contents

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

A vector-valued function (or, vector function) is a parametric function whose domain is a set of real numbers and whose range is a set of vectors. This can be defined parametrically as:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \tag{13.1}$$

The limit of this function is taken as:

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle \tag{13.2}$$

13.1.1 Continuity

A vector function $\vec{r}$ is continous at a if:

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a) \tag{13.3}$$

Suppose that $\vec{r}(t)$ is continuous over all points in I. Therefore, a curve C (a space curve) is defined to be all points (x, y, z) that satisfy those parametric equations above.

13.2 Derivatives and integrals of vector functions

13.3 Derivatives

The derivative of a vector-valued function is given for:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \tag{13.4}$$

as:

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t) \rangle \tag{13.5}$$

And follows the following differentiation rules given that c is a scalar constant, f is a real-valued function, and u and v are vector-valued functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{v'}(t) + \vec{u'}(t) \tag{13.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[c\vec{u}(t)\right] = c\vec{u'}(t) \tag{13.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\vec{v}(t)\right] = f'(t)\vec{v}(t) + f(t)\vec{v'}(t) \tag{13.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u'}(t) \cdot \vec{v}(t) + \vec{v'}(t) \cdot \vec{u}(t) \tag{13.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u'}(t) \times \vec{v}(t) + \vec{v'}(t) \times \vec{u}(t) \tag{13.10}$$

13.3.1 Integrals

The integral of a vector-valued function $r(t)$ as given above, is defined as:

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \langle \int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t, \int_a^b h(t)\,\mathrm{d}t \rangle \tag{13.11}$$

13.4 Arc Length & Curvature

The length of a space curve in $\mathbb{R}^3$ given as $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is defined to be:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,\mathrm{d}t \tag{13.12}$$

Notice, however, that:

$$\vec{r}'(t) = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \tag{13.13}$$

therefore:

$$L = \int_a^b \vec{r'}(t)\,\mathrm{d}t \tag{13.14}$$

13.4.1 Re-parameterization

Let the arc-length function from a to t be defined as:

$$s(t) = \int_a^t \vec{r'}(u)\,\mathrm{d}u \tag{13.15}$$

Given this, it is often useful to re-parameterize the function $\vec{r}(t)$ in terms of arc-length, s, not time elapsed, t. To do so, note:

$$t = t(s) \to r(t) = r(t(s)) \to r(s) \tag{13.16}$$

13.4.2 Curvature

Let the unit tangent vector for a curve C be given as:

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} \tag{13.17}$$

The curvature, κ of a given space curve C is given as:

$$\kappa = \left|\frac{\mathrm{d}T}{\mathrm{d}s}\right| \tag{13.18}$$

or defined by The Chain Rule, as:

$$\kappa = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3} \tag{13.19}$$

Let the unit normal vector be defined as:

$$\vec{N} = \frac{1}{\vec{T'}(t)} \vec{T'}(t) \tag{13.20}$$

The binormal vector, orthogonal to both the unit tangent and unit normal vectors is given as:

$$\vec{B} = \vec{T} \times \vec{N} \tag{13.21}$$

Normal Plane

This plane contains all lines orthogonal to the space curve C at $P(x, y, z)$. It is thought to be the "worst" approximation to the curve at that point. It contains the binormal and normal vectors, thus:

$$T_1(x - x_1) + T_2(y - y_1) + T_3(z - z_1) = 0 \tag{13.22}$$

Osculating Plane

This plane is closest to the space curve C at $P(x, y, z)$. It is thought to be the "best" approximation to the curve at that point. It contains the tangent and normal vectors, and thus:

$$B_1(x - x_1) + B_2(y - y_1) + B_3(z - z_1) = 0 \tag{13.23}$$

Osculating Circle

This circle is defined for $\mathbb{R}^2$ curves as the closest circle to a function $x = f(t)$, $y = g(t)$ for which the circle contains the same point of intersection, tangency, and curvature.

$$(x(t) - \frac{1}{\kappa(t)})^2 + (y(t) - \frac{1}{\kappa(t)})^2 = \frac{1}{\kappa(t)^2} \tag{13.24}$$

13.5 Motion in Space: Velocity & Acceleration

Recall that, for a position function:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \tag{13.25}$$

Its velocity function is defined as:

$$\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t) = \langle x'(t), y'(t), z'(t) \rangle \tag{13.26}$$

Note: the magnitude of velocity v is defined as:

$$v = |\vec{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \tag{13.27}$$

And its acceleration function is defined as:

$$\vec{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) = \langle x''(t), y''(t), z''(t) \rangle \tag{13.28}$$

Similarly:

$$\vec{v}(t) = \int \vec{a}(t)\,\mathrm{d}t \tag{13.29}$$

as well:

$$\vec{r}(t) = \int \vec{v}(t)\,\mathrm{d}t \tag{13.30}$$

Recall, however, that integrating produces a constant of integration in each dimension (C_1, C_2, etc.) and must be determined to be a linear scalar quantity at each step.

13.5.1 Newton's Second Law of Motion

Recall that Newton's Second Law of Motion is stated as:

$$\vec{F} = m\vec{a} \tag{13.31}$$

which indicates that $\vec{a}$, m, or $\vec{F}$ can be solved for when values for the other two are known.

13.6 Tangential and Normal Components

It is useful to express the components of acceleration ($\vec{a}$) in terms of a scalar quantity along the tangent vector ($\vec{T}$) and the normal vector ($\vec{N}$):

$$a_T = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{|\vec{r'}(t)|} \tag{13.32}$$

$$a_N = \frac{\vec{r'}(t) \times \vec{r''}(t)}{|\vec{r'}(t)|} \tag{13.33}$$

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

A function f fo two variables assigns each (x, y) pair to an output in the range of f, such that:

$$\{f(x, y)|(x, y) \in D\} \tag{14.1}$$

additionally:

$$\{(x, y) \subseteq \mathbb{R}^2, f(x, y) \subseteq \mathbb{R}\} \tag{14.2}$$

14.1.1 Linear Functions of Three Variables

A linear functions is defined as:

$$Ax + By + C = z \tag{14.3}$$

14.1.2 Level Curves

A level curve for $f(x, y)$ at k (read: the k-th level curve for f) is given as:

$$\{(x, y) \subseteq \mathbb{R}^2|f(x, y) = k\} \tag{14.4}$$

or, alternatively:

$$f(x, y) = k \tag{14.5}$$

14.1.3 Higher Dimensions

Functions can take place in arbitrarily-high dimensions. For instance, the function f:

$$f(x_1, x_2, \ldots, x_n) \tag{14.6}$$

maps inputs in the domain set to the n-tuple:

$$(x_1, x_2, \ldots, x_n, z) \tag{14.7}$$

such that:

$$\{x_1, x_2, \ldots, x_n \subseteq \mathbb{R}^n | f(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1}\} \tag{14.8}$$

Or, in other words, a function f increases dimensionality from its input space of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ in its output space.

14.2 Partial Derivatives

Derivates for higher dimension functions are taken one variable at a time. The algorithm is as follows:

1. To differentiate a multivariate function, take the remaining free variables and for each:

2. Differentiate the function with respect to that variable, leaving all other expressions as constants.

3. Repeat until there are no more remaining free variables.

There are several ways to notate this operation:

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \tag{14.9}$$

14.3 Tangent Planes and Linear Approximations

Here, we present an analogue of approximate curves in $\mathbb{R}^2$ with a linear approximation to approximating surfaces in $\mathbb{R}^3$ with a tangent plane that behaves similarly to the function $z = f(x, y)$ "near" point $P_0(x_0, y_0, f(x_0, y_0))$.

14.3.1 Tangent Planes

Note that the plane will contain a vector normal to the surface at the point ($\vec{n} = \langle n_1, n_2, n_3 \rangle$), and the point itself ($P_0(x_0, y_0, f(x_0, y_0))$), of the form:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \tag{14.10}$$

Let $\vec{r}_x(t)$ be the path around the curve along the x-axis, as follows:

$$\vec{r}_x(t) = \langle t, y_0, f(t, y_0) \rangle \tag{14.11}$$

Further, let $\vec{r}_y(t)$ be the path around the curve along the x-axis, as follows:

$$\vec{r}_y(t) = \langle x_0, t, f(x_0, t) \rangle \tag{14.12}$$

Note that the first derivatives of these functions will yield a set of tangent vectors along each coordinate axis for the surface in question:

$$\vec{r}_x'(t) = \langle 1, 0, f_x(t, y_0) \rangle \tag{14.13}$$

$$\vec{r}_y'(t) = \langle 0, 1, f_y(x_0, t) \rangle \tag{14.14}$$

Further note that their cross-product will yield a vector tangent to the surface at the point $P_0(x_0, y_0, f(x_0, y_0))$:

$$\vec{n} = \vec{r}_x' \times \vec{r}_y' = \langle f_x(t, y_0), f_y(x_0, t), 1 \rangle \tag{14.15}$$

Therefore, the plane of tangency for the surface $z = f(x, y)$ in $\mathbb{R}^3$ is given as:

$$f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) - (z - z_0) = 0 \tag{14.16}$$

14.3.2 Linear Approximations

Let the tangent plane be used to approximate the surface $z = f(x, y)$ in $\mathbb{R}^3$.

Expand the equation above:

$$\begin{aligned} f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) - (z - z_0) &= 0 \\ f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) + z_0 &= z \end{aligned}$$

Denote z to be the linear approximation ($L(x,y)$) as follows:

$$L(x,y) = z_0 + f_x(t, y_0)(x - x_0) + f_y(x_0, t)(y - y_0) \tag{14.17}$$

where:

$$L(x,y) \simeq f(x,y) \tag{14.18}$$

and is especially well-behaved around input points near $P(x_0, y_0)$.

14.3.3 Differentiability

Thm. If the partial derivatives f_x and f_y exist near (a,b) and are continuous at a, and b, then f is differentiable at (a,b).

14.3.4 Total differentials

Let $\mathrm{d}z$ be the approximated change in the function $f(x,y)$ along a change $\Delta x = \mathrm{d}x$, and $\Delta y = \mathrm{d}y$

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y \tag{14.19}$$

Or, for equations represented in $\mathbb{R}^n$:

$$\mathrm{d}z = \sum_{i=0}^{n} \frac{\partial z}{\partial d_i}\,\mathrm{d}d_i \tag{14.20}$$

14.4 Optimization

Let the critical points of $z = f(x,y)$ be defined as the places where either f_x and f_y are concave down, concave up, or one of each.

14.4.1 Critical points of f

Let the set of all critical points be denoted:

$$\{(x_0, y_0) : f_x(x_0, y_0) = f_y(x_0, y_0) = 0\} \tag{14.21}$$

or where f_x, f_y evaluated at (x_0, y_0) is undefined.

14.4.2 2nd derivative test

For each critical point, let its classification be defined as:

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} \tag{14.22}$$

$D(x_0, y_0) > 0$ $f(x,y)$ has a local maxima or minima at (x_0, y_0) if f_{xx} is negative, or positive, respectively.

$D(x_0, y_0) < 0$ $f(x,y)$ has a saddle point at (x_0, y_0).S

$D(x_0, y_0) = 0$ Second derivative test has failed.

14.4.3 Extreme Value Theorem

If $f(x,y)$ is defined on a closed, bounded set $D \subseteq \mathbb{R}^2$, then f attains a maximum value at $f(x_1, y_1)$ at $(x_1, y_1) \in D$, and a minimum value at $f(x_2, y_2)$ at $(x_2, y_2) \in D$.

1. Find all critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. Let the largest value of the union of the above be the absolute (global) maximum of the function f, and the smallest be its minimum.

Chapter 15

Multiple Integrals

15.1 Double Integrals Over Rectangles

Recall the definition of an integral over a range of points in $\mathbb{R}^2$:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \tag{15.1}$$

In $\mathbb{R}^3$, the definition is as follows:

Suppose you have a function $z = f(x,y)$ in $\mathbb{R}^3$, and a closed rectangle in the domain of that function:

$$R = \{(x,y) \in \mathbb{R}^2 | x \in [a,b] \wedge y \in [c,d]\} \tag{15.2}$$

Further suppose that you have a surface S defined in the area above or below that region R:

$$S = \{(x,y,z) \in \mathbb{R}^3 | z \in [0, f(x,y)] \wedge (x,y) \in \mathbb{R}^2\} \tag{15.3}$$

Define the i-th and j-th rectangular subdivision of R to be:

$$R_{ij} = \{(x,y) | x \in [x_{i-1}, x_i] \wedge y \in [y_{j-1}, y_j]\} \tag{15.4}$$

such that:

$$\Delta A = \Delta x \Delta y \tag{15.5}$$

Further denote a sample within the i-th and j-th region of R to be: R_{ij}^*

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
$$V = \iint\limits_R f(x,y)\,\mathrm{d}A$$

15.1.1 The Midpoint Rule

Similar to how the midpoints of Riemann sums could be used to approximate the area under a curve for functions in $\mathbb{R}^2$, the midpoint of the i-th and $j-th$ sub-rectangle (denoted $(\bar{x}_i, \bar{y}_i)$) can be used to approximate the volume enclosed in that rectangle:

$$\iint\limits_R f(x,y)\,\mathrm{d}A \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A \tag{15.6}$$

15.1.2 The Average Value Theorem

The average value for a surface in $R \subset \mathbb{R}^2$ can be calculated as the volume of the double-integral divided by the area (denoted as the function $A(R)$) of the area R:

$$f_{avg} = \frac{1}{A(R)}\iint\limits_R f(x,y)\,\mathrm{d}A \tag{15.7}$$

15.2 Iterated Integrals

When presented with a double integral, the contained integrals may be *partially integrated* until the integral is complete:

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_a^b \left[\int_c^d f(x,y)\,\mathrm{d}y\right] \tag{15.8}$$

Where the integral taken with respect to dy is evaluated as if any terms of x were constant. The result of this (with the unevaluated "constant" x-terms) is then integrated with respect to dx.

15.2.1 Fubini's Theorem

Presented without proof, Fubini's Theorem states that the integral of multiple dimensions can be taken in any order, similar to Clairaut's Theorem states that functions of higher dimensions can be differentiated in any order.

$$\begin{aligned}\iint\limits_R f(x,y)\,\mathrm{d}A &= \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

15.3 Double Integrals over General Regions

15.3.1 Type I Regions

To take the integral of $z = f(x,y)$ in a region D bounded by two continuous functions of x, say: $g_1(x)$, $g_2(x)$, apply the following:

Let:

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in \mathrm{dom}(f) \\ 0 & \end{cases} \tag{15.9}$$

therefore:

$$\iint\limits_D f(x,y)\,\mathrm{d}A = \iint\limits_R F(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{15.10}$$

15.3.2 Type II Regions

To take the integral of $z = f(x,y)$ in a region D bounded by two continuous functions of y, say: $g_1(y)$, $g_2(y)$, apply the following:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{15.11}$$

15.3.3 Properties of Double Integrals

Double integrals are associative over summations of inner functions $f_i(x,y)$:

$$\iint_D [f(x,y) + g(x,y)]\,\mathrm{d}A = \iint_D f(x,y)\,\mathrm{d}A + \iint_D g(x,y)\,\mathrm{d}A \tag{15.12}$$

Double integrals are associative over constants c in the inner functions:

$$\iint_D cf(x,y)\,\mathrm{d}A = c\iint_D f(x,y)\,\mathrm{d}A \tag{15.13}$$

Double integrals are commutative over sub-divisions of the region of integration D. Let $D = D_1 \cup D_2$:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A \tag{15.14}$$

Double integrals also hold the following property:

$$\iint_D c\,\mathrm{d}A = cA(D) \tag{15.15}$$

where $A(D)$ is the area of the region D.

Finally, if $m \leq f(x,y) \leq M \forall (x,y) \in D$:

$$mA(D) \leq \iint f(x,y)\,\mathrm{d}A \leq MA(D) \tag{15.16}$$

To determine m and M, use methods from the previous chapter to optimize the function $z = f(x,y)$ over the given domain, D.

Double integrals can be used to find the average value (denoted for $z = f(x,y)$ as f_{avg}), as:

$$f_{avg} = \frac{1}{A(D)} \iint\limits_D f(x,y)\,\mathrm{d}A \tag{15.17}$$

where $A(D)$ is the area given by D, which could be a rectangle, triangle, polar region, or Calculus II function that must itself be integrated.

15.3.4 Order of Integration

It is sometimes useful to reverse the order of integration from $\mathrm{d}x\,\mathrm{d}y$ to $\mathrm{d}y\,\mathrm{d}x$. In order to do so, consider the original region of integration. Rewrite functions of x as functions of y, and vice-versa. Rewrite the integral with the functional limits at the innermost position, and reverse the order of differentials. Integrate.

15.4 Double Integrals in Polar Coordinates

In certain cases, the above "Type I" and "Type II" integrals are not ideal tools to integrate complex general regions. Polar coordinates can be a fit for this instead:

15.4.1 Polar Rectangles

A polar rectangle is a 2-dimensional donut over any arbitrary angle region θ. For instance:

$$R = \{(r,\theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\} \tag{15.18}$$

15.4.2 Converting Cartesian Functions

Recall the following:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Therefore, the polar conversion of $z = f(x, y)$ to $z = f(r, \theta)$ is given as follows:

$$f(x, y) = f(r\cos\theta, r\sin\theta) \tag{15.19}$$

When f is continuous over a polar rectangle $0 \le a \le r \le b$, and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$:

$$\iint\limits_D f(x, y)\,\mathrm{d}A = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta \tag{15.20}$$

15.5 Applications of Double Integrals

15.5.1 Density

Recall that the "instantaneous" density for a surface in $\mathbb{R}^3$ is denoted as:

$$\rho(x, y) \tag{15.21}$$

Where the quantity is mass per unit-volume.

Therefore, to calculate the mass m of a surface in $\mathbb{R}^3$ over the sub-domain D, apply:

$$m_D = \iint\limits_D \rho(x, y)\,\mathrm{d}A \tag{15.22}$$

15.5.2 Moments

Recall that a "moment" is defined as the weighted sum of all particles in a surface along a directed axis.

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D y\rho(x, y)\,\mathrm{d}A \tag{15.23}$$

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D x\rho(x, y)\,\mathrm{d}A \tag{15.24}$$

Therefore, the center of mass $(\bar{x}, \bar{y})$ is given to be:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{1}{m} \iint_D x\rho(x, y)\, \mathrm{d}A, \frac{1}{m} \iint_D y\rho(x, y)\, \mathrm{d}A\right) \quad (15.25)$$