MATH 307 A, Clancy

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Chapter 1

Introduction

1.1 Introduction

Many equations involve quantities or functions that pertain to the *rate* at which processes occur. Such equations are called *differential equations*.

1.1.1 Classifications

There are two important classifications of differential equations:

Ordinary Differential Equations Equations involving a derivative of one or more functions, but only with respect to a single independent variable (as opposed to a partial derivative, that can be taken with respect to multiple independent variables).

Partial Differential Equation Equations that involve partial derivatives, or derivatives that can be taken with respect to multiple independent variables.

1.1.2 Order

Define the *order* of a differential equation to be the maximum order of each differential component of the equation.

1.1.3 Linearity

Define a linear differential equation F to be one who is a linear function in y of all of its various $y^{(n)}$ derivatives. Define a non-linear differential equation to be one that is not.

In other terms, a linear differential equation can be written as:

$$\sum_{i=0}^{n} a_i(x) \cdot y^{(n-i)}$$

Where each a_i is a function of x (linear or not), but does not involve y. Note that $y^{(n-i)}$ denotes the $n - i$-th derivative of y.

1.1.4 Separability

Define a separable differential equation to be one for which there exists a function $f(t)$, such that:

$$\frac{dy}{dt} = f(t) \cdot g(y)$$

When this is linear, this is called a *separable, first-order linear differential equation*. To solve equations of this form, apply the following process:

1. Begin by rewriting as:
$$\frac{1}{g(y)} dy = f(t) dt$$
2. Then, integrate both sides as:
$$\int \frac{1}{g(y)} dy = \int f(t) dt$$
3. And, if possible, solve for y. If y is not easily found, then leave it as a "nice" implicit solution.

Example

Here is an example differential equation, that we wish to solve for y.

$$\frac{dy}{dx} = 3x^2y^2$$

Begin by separating like terms and their differentials:

$$\frac{1}{3x^2}\ dy = 3x^2\ dx$$

Then, integrate both sides:

$$\begin{aligned} \int y^{-2}\ dy &= \int 3x^2\ dx \\ -y^{-1} &= 3 \cdot \int x^2\ dx \\ &= \frac{\cancel{3}}{\cancel{3}} x^3 + C_1 \\ &= x^3 + C_1 \end{aligned}$$

Finally simplifying to:

$$y = \frac{1}{x^3 + C_2}$$

Chapter 2

First Order Differential Equations

2.1 Initial Value Problem

Consider a differential equation of the form:

$$\frac{dy}{dt} = ay - b$$

Where both a and b are constants. Consider that we are also privy to the fact that $y(0) = y_0$. This problem, and problems like it are called "Initial Value Problem(s)", or I.V.P., for short.

To solve an I.V.P., we first prove the general solution, and then state it. We begin as follows:

$$\begin{aligned}\frac{dy}{dt} &= ay - b \\ &= a(y - \frac{b}{a})\end{aligned}$$

We then separate the variables so that the equation is integrable:

$$\int \frac{dy}{y - \frac{b}{a}} = \int a \; dt$$

Simplifying both sides, we obtain:

$$\ln \left| y - \frac{b}{a} \right| = at + C_1$$

We exponentiate both sides (letting $A = e^{C_1}$):

$$\left| y - \frac{b}{a} \right| = Ae^{at}$$

Going further, we obtain:

$$y - \frac{b}{a} = \pm Ae^{at}$$

Finally:

$$y = Ae^{at} + \frac{b}{a}$$

We can then solve the initial value constraint ($y(0) = y_0$) by:

$$y_0 = Ae^{a \cdot 0} + \frac{b}{a}$$

To obtain:

$$y(t) = \left(y_0 - \frac{b}{a} \right) e^{at} + \frac{b}{a}.$$

2.2 Euler's Method

Euler's method gives us an approximation for a function f based on a step function, and its derivative, $\frac{d}{dt}f$. We use the derivative of the function in order to provide an approximation by considering small changes in t along tangent lines to approximate the integral curve.

$$y_{n+1} = y_n + \frac{d}{dt}y \bigg|_{\substack{t=t_n \\ y=y_n}} \cdot (t_{n+1} - t_n)$$

2.3 Linear Equations

We describe a method of solving linear equations known as "Integrating Factors". Recall that the specific form of a linear equation thus far has been:

$$\frac{dy}{dt} = -ay + b$$

We expand this to be more general, defining a *first order linear differential equation* to be:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Or alternatively:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

2.3.1 Direct Integration

Some equations of the later form can be solved by direct integration. To verify this, divide the equation by $P(t)$, and factor to attempt to separate the t's and y's on either side of the equation. Then, if possible, attempt to integrate both sides.

Consider the general form of the equation (representing a separable, first-order linear ordinary differential equation):

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

We write it in separated form as:

$$M(x)dx + N(y)dy = 0$$

Then solve and integrate both sides. To find ϕ, we solve for $y(0) = y_0$.

2.3.2 Integrating Factor

Unfortunately, in the overwhelming majority of differential equations of this classification, the above technique cannot be applied. Instead, we find a function $\mu(t)$ which we call an *integrating factor*.

Our goal is to find a function $\mu(t)$, which when treated as follows:

$$\mu(t)P(t)\frac{dy}{dt} + \mu(t)Q(t)y = \mu(t)G(t)$$

Consider a general equation of the form $\frac{dy}{dt} + p(t)y = g(t)$. In general, we go as follows:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply both sides of the equation by our integrating factor, $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

Solve for $\mu(t)$ so that $\frac{d}{dt}[\mu(t)y]$ is equal to the left-hand side of the above equation:

$$\frac{d}{dt}[\mu(t)y] = \underbrace{\frac{d\mu(t)}{dt}y + \frac{dy}{dt}\mu(t)}_{\text{Product Rule}} = \overbrace{\mu(t)\frac{dy}{dt} + \mu(t)p(t)y}^{\text{Left-hand side}}$$

Provided that:

$$\frac{d\mu(t)}{dt}y = \mu(t)p(t)y \qquad \Longleftrightarrow \qquad \frac{d\mu(t)}{dt} = \mu(t)p(t)$$

Assume that $\mu(t) > 0$. Then we have:

$$\begin{aligned}
\frac{\frac{d\mu(t)}{dt}}{\mu(t)} &= p(t) \\
\frac{d\mu(t)}{\mu(t)} &= p(t)dt \\
\int \frac{d\mu(t)}{\mu(t)} &= \int p(t)dt \\
\ln|\mu(t)| &= \int p(t)dt + k \\
\mu(t) &= e^{\int p(t)dt}
\end{aligned}$$

Note that we now have (per above) that:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t)$$

Therefore:

$$\mu(t)y = \int \mu(t)g(t)dt$$

Thus, the general solution of $\frac{dy}{dt} + p(t)y = g(t)$ is given as:

$$\boxed{y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\, ds + C\right]}$$

2.4 Modeling

Mathematical models are a useful tool to approximate real-world processes. Often times, differential equations are a particularly useful model. When creating a model of a real-world process and/or event, we must note the shortcomings of that model, and *analyze* it against our actual observations. Nonetheless, these models are a useful tool for approximation. We describe several general examples here:

2.4.1 Mixing

Suppose there is a tank that at time t_0 contains volume Q_0 of salt. We write $\frac{dQ}{dt}$ (the rate of change of salt in the tank with respect to time) as:

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Consider "rate in". We are given a concentration of salt flowing into the tank (in mass per volume) and a flow rate (in volume per time). We determine the "rate in" as the product of the two.

Consider "rate out". We are given that the tank is stirred continually (i.e., that the concentration of salt is uniform throughout the tank). We are given that the rate of flow out is r (in volume per time), and know the volume of the tank. Thus, the "rate out" is given to be the product of the flow rate, the total amount of salt, divided by the volume of the tank.

We solve this by the method of integrating factors, as (according to example 2.3.1 in the book):

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \qquad \text{and} \qquad Q(0) = Q_0$$

Rewriting into the standard form of a linear equation, we obtain:

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}$$

Therefore:

$$\mu(t) = e^{\frac{rt}{100}} \qquad \text{and} \qquad Q(t) = 25 + (Q_0 - 25)e^{-\frac{rt}{100}}$$

2.4.2 Compound Interest

Suppose that you have a bank account that pays an interest rate annually at r. The value $S(t)$ of your investment is determined continually. Thus, the rate at which your investment size changes is given as:

$$\frac{dS}{dt} = rS \qquad \text{and} \qquad S(0) = S_0$$

We can therefore see that the equation is both linear and separable. Thus:

$$S(t) = S_0 e^{rt}$$

2.4.3 Escape Velocity

Suppose that we project a body of mass away from the earth at $v(0) = v_0$. We find the maximum height ξ that the body will reach, and the minimum such v_0 to ensure that the body does not return to earth (read: escape velocity).

Note that the force acting on a body as a function of its position and earth's gravity is given as:

$$F(x) = m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$$

We note also that:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

Therefore,

$$F(x) = \cancel{m}v\frac{dv}{dx} = -\frac{\cancel{m}gR^2}{(R+x)^2}$$

We note that this equation is separable, but not linear, so it is solved by:

$$\frac{1}{2}v^2 = \frac{gR^2}{(R+x)} + C \qquad \text{and} \qquad v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$$

To solve for its maximum altitude, $x = \xi$, set $v = 0$, and $x = \xi$, and obtain:

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}$$

To find the initial velocity, $v = v_0$, required to lift the body to height $x = \xi$, we solve:

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}$$

And to find the escape velocity, we note that:

$$v_e = \lim_{\xi\to\infty}\sqrt{2gR\frac{\xi}{R+\xi}} = \sqrt{2gR}$$

2.5 Autonomous Equations

Another class of differential equation is one where the independent variable does not explicitly appear. Such an equation is called *autonomous*:

$$\frac{dy}{dt} = f(y)$$

2.5.1 Exponential Population Dynamics

Suppose that we assume a growth rate, r, proportional to the size of the population at a given time, t, $y(t)$. Let $y = \phi(t)$ be the size of the population at a given time. Therefore,

$$\frac{dy}{dt} = ry$$

Subject to the initial condition $y(0) = y_0$, we obtain that:

$$y = y_0 e^{rt}$$

2.5.2 Logistic Population Dynamics

Suppose instead that the rate of growth itself depends on the population, such that $r = h(y)$. Then:

$$\frac{dy}{dt} = h(y)y$$

We wish to use a $h(y) \simeq r > 0$. We use the Verhulst equation (oftentimes, "the logistic equation") such that:

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

Where $K = \frac{r}{a}$, and r is the intrinsic growth rate.

We seek simple, constant functions for y, such that $y = \phi_1 = 0$ and $y = \phi_2 = K$. We classify these solutions as *equilibrium solutions*. We further describe them by:

Stable equilibrium solution An equilibrium solution that is converged upon from both sides.

Semi-stable equilibrium solution An equilibrium solution that is converged upon from one, but not both sides.

Unstable equilibrium solution An equilibrium solution that is diverged upon from both sides.

A *phase line* is used as an accompanying context to the y-axis to, for each equilibria, show whether we are converging towards or away from that solution.

If we wish to solve the specific equation, we go as follows:

$$\frac{dy}{(1 - \frac{y}{K})y} = r\ dt$$

And perform a partial fraction expansion on the left-hand side as:

$$\left(\frac{1}{y} + \frac{\frac{1}{K}}{1 - \frac{y}{K}}\right) \, dy = r \, dt$$

Therefore,

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$