MATH 307 A (Final), Clancy

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Chapter 1

Introduction

1.1 Introduction

Many equations involve quantities or functions that pertain to the *rate* at which processes occur. Such equations are called *differential equations*.

1.1.1 Classifications

There are two important classifications of differential equations:

Ordinary Differential Equations Equations involving a derivative of one or more functions, but only with respect to a single independent variable (as opposed to a partial derivative, that can be taken with respect to multiple independent variables).

Partial Differential Equation Equations that involve partial derivatives, or derivatives that can be taken with respect to multiple independent variables.

1.1.2 Order

Define the *order* of a differential equation to be the maximum order of each differential component of the equation.

1.1.3 Linearity

Define a linear differential equation F to be one who is a linear function in y of all of its various $y^{(n)}$ derivatives. Define a non-linear differential equation to be one that is not.

In other terms, a linear differential equation can be written as:

$$\sum_{i=0}^{n} a_i(x) \cdot y^{(n-i)}$$

Where each a_i is a function of x (linear or not), but does not involve y. Note that $y^{(n-i)}$ denotes the $n - i$-th derivative of y.

1.1.4 Separability

Define a separable differential equation to be one for which there exists a function $f(t)$, such that:

$$\frac{dy}{dt} = f(t) \cdot g(y)$$

When this is linear, this is called a *separable, first-order linear differential equation*. To solve equations of this form, apply the following process:

1. Begin by rewriting as:
$$\frac{1}{g(y)} dy = f(t) dt$$
2. Then, integrate both sides as:
$$\int \frac{1}{g(y)} dy = \int f(t) dt$$
3. And, if possible, solve for y. If y is not easily found, then leave it as a “nice” implicit solution.

Example

Here is an example differential equation, that we wish to solve for y.

$$\frac{dy}{dx} = 3x^2 y^2$$

Begin by separating like terms and their differentials:

$$\frac{1}{3x^2}\ dy = 3x^2\ dx$$

Then, integrate both sides:

$$\begin{aligned} \int y^{-2}\ dy &= \int 3x^2\ dx \\ -y^{-1} &= 3 \cdot \int x^2\ dx \\ &= \frac{\cancel{3}}{\cancel{3}} x^3 + C_1 \\ &= x^3 + C_1 \end{aligned}$$

Finally simplifying to:

$$y = \frac{1}{x^3 + C_2}$$

Chapter 2

First Order Differential Equations

2.1 Initial Value Problem

Consider a differential equation of the form:

$$\frac{dy}{dt} = ay - b$$

Where both a and b are constants. Consider that we are also privy to the fact that $y(0) = y_0$. This problem, and problems like it are called "Initial Value Problem(s)", or I.V.P., for short.

To solve an I.V.P., we first prove the general solution, and then state it. We begin as follows:

$$\begin{aligned}\frac{dy}{dt} &= ay - b \\ &= a(y - \frac{b}{a})\end{aligned}$$

We then separate the variables so that the equation is integrable:

$$\int \frac{dy}{y - \frac{b}{a}} = \int a \; dt$$

Simplifying both sides, we obtain:

$$\ln\left|y - \frac{b}{a}\right| = at + C_1$$

We exponentiate both sides (letting $A = e^{C_1}$):

$$\left|y - \frac{b}{a}\right| = Ae^{at}$$

Going further, we obtain:

$$y - \frac{b}{a} = \pm Ae^{at}$$

Finally:

$$y = Ae^{at} + \frac{b}{a}$$

We can then solve the initial value constraint ($y(0) = y_0$) by:

$$y_0 = Ae^{a \cdot 0} + \frac{b}{a}$$

To obtain:

$$y(t) = \left(y_0 - \frac{b}{a}\right) e^{at} + \frac{b}{a}.$$

2.2 Euler's Method

Euler's method gives us an approximation for a function f based on a step function, and its derivative, $\frac{d}{dt}f$. We use the derivative of the function in order to provide an approximation by considering small changes in t along tangent lines to approximate the integral curve.

$$y_{n+1} = y_n + \frac{d}{dt}y\bigg|_{\substack{t=t_n \\ y=y_n}} \cdot (t_{n+1} - t_n)$$

2.3 Linear Equations

We describe a method of solving linear equations known as "Integrating Factors". Recall that the specific form of a linear equation thus far has been:

$$\frac{dy}{dt} = -ay + b$$

We expand this to be more general, defining a *first order linear differential equation* to be:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Or alternatively:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

2.3.1 Direct Integration

Some equations of the later form can be solved by direct integration. To verify this, divide the equation by $P(t)$, and factor to attempt to separate the t's and y's on either side of the equation. Then, if possible, attempt to integrate both sides.

Consider the general form of the equation (representing a separable, first-order linear ordinary differential equation):

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

We write it in separated form as:

$$M(x)dx + N(y)dy = 0$$

Then solve and integrate both sides. To find ϕ, we solve for $y(0) = y_0$.

2.3.2 Integrating Factor

Unfortunately, in the overwhelming majority of differential equations of this classification, the above technique cannot be applied. Instead, we find a function $\mu(t)$ which we call an *integrating factor*.

Our goal is to find a function $\mu(t)$, which when treated as follows:

$$\mu(t)P(t)\frac{dy}{dt} + \mu(t)Q(t)y = \mu(t)G(t)$$

Consider a general equation of the form $\frac{dy}{dt} + p(t)y = g(t)$. In general, we go as follows:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply both sides of the equation by our integrating factor, $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

Solve for $\mu(t)$ so that $\frac{d}{dt}[\mu(t)y]$ is equal to the left-hand side of the above equation:

$$\frac{d}{dt}[\mu(t)y] = \underbrace{\frac{d\mu(t)}{dt}y + \frac{dy}{dt}\mu(t)}_{\text{Product Rule}} = \overbrace{\mu(t)\frac{dy}{dt} + \mu(t)p(t)y}^{\text{Left-hand side}}$$

Provided that:

$$\frac{d\mu(t)}{dt}y = \mu(t)p(t)y \qquad \Longleftrightarrow \qquad \frac{d\mu(t)}{dt} = \mu(t)p(t)$$

Assume that $\mu(t) > 0$. Then we have:

$$\begin{aligned}
\frac{\frac{d\mu(t)}{dt}}{\mu(t)} &= p(t) \\
\frac{d\mu(t)}{\mu(t)} &= p(t)dt \\
\int \frac{d\mu(t)}{\mu(t)} &= \int p(t)dt \\
\ln|\mu(t)| &= \int p(t)dt + k \\
\mu(t) &= e^{\int p(t)dt}
\end{aligned}$$

Note that we now have (per above) that:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t)$$

Therefore:

$$\mu(t)y = \int \mu(t)g(t)dt$$

Thus, the general solution of $\frac{dy}{dt} + p(t)y = g(t)$ is given as:

$$\boxed{y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\, ds + C\right]}$$

2.4 Modeling

Mathematical models are a useful tool to approximate real-world processes. Often times, differential equations are a particularly useful model. When creating a model of a real-world process and/or event, we must note the shortcomings of that model, and *analyze* it against our actual observations. Nonetheless, these models are a useful tool for approximation. We describe several general examples here:

2.4.1 Mixing

Suppose there is a tank that at time t_0 contains volume Q_0 of salt. We write $\frac{dQ}{dt}$ (the rate of change of salt in the tank with respect to time) as:

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Consider "rate in". We are given a concentration of salt flowing into the tank (in mass per volume) and a flow rate (in volume per time). We determine the "rate in" as the product of the two.

Consider "rate out". We are given that the tank is stirred continually (i.e., that the concentration of salt is uniform throughout the tank). We are given that the rate of flow out is r (in volume per time), and know the volume of the tank. Thus, the "rate out" is given to be the product of the flow rate, the total amount of salt, divided by the volume of the tank.

We solve this by the method of integrating factors, as (according to example 2.3.1 in the book):

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \qquad \text{and} \qquad Q(0) = Q_0$$

Rewriting into the standard form of a linear equation, we obtain:

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}$$

Therefore:

$$\mu(t) = e^{\frac{rt}{100}} \qquad \text{and} \qquad Q(t) = 25 + (Q_0 - 25)e^{-\frac{rt}{100}}$$

2.4.2 Compound Interest

Suppose that you have a bank account that pays an interest rate annually at r. The value $S(t)$ of your investment is determined continually. Thus, the rate at which your investment size changes is given as:

$$\frac{dS}{dt} = rS \qquad \text{and} \qquad S(0) = S_0$$

We can therefore see that the equation is both linear and separable. Thus:

$$S(t) = S_0 e^{rt}$$

2.4.3 Escape Velocity

Suppose that we project a body of mass away from the earth at $v(0) = v_0$. We find the maximum height ξ that the body will reach, and the minimum such v_0 to ensure that the body does not return to earth (read: escape velocity).

Note that the force acting on a body as a function of its position and earth's gravity is given as:

$$F(x) = m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$$

We note also that:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

Therefore,

$$F(x) = \cancel{m}v\frac{dv}{dx} = -\frac{\cancel{m}gR^2}{(R+x)^2}$$

We note that this equation is separable, but not linear, so it is solved by:

$$\frac{1}{2}v^2 = \frac{gR^2}{(R+x)} + C \qquad \text{and} \qquad v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$$

To solve for its maximum altitude, $x = \xi$, set $v = 0$, and $x = \xi$, and obtain:

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}$$

To find the initial velocity, $v = v_0$, required to lift the body to height $x = \xi$, we solve:

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}$$

And to find the escape velocity, we note that:

$$v_e = \lim_{\xi\to\infty}\sqrt{2gR\frac{\xi}{R+\xi}} = \sqrt{2gR}$$

2.5 Autonomous Equations

Another class of differential equation is one where the independent variable does not explicitly appear. Such an equation is called *autonomous*:

$$\frac{dy}{dt} = f(y)$$

2.5.1 Exponential Population Dynamics

Suppose that we assume a growth rate, r, proportional to the size of the population at a given time, t, $y(t)$. Let $y = \phi(t)$ be the size of the population at a given time. Therefore,

$$\frac{dy}{dt} = ry$$

Subject to the initial condition $y(0) = y_0$, we obtain that:

$$y = y_0 e^{rt}$$

2.5.2 Logistic Population Dynamics

Suppose instead that the rate of growth itself depends on the population, such that $r = h(y)$. Then:

$$\frac{dy}{dt} = h(y)y$$

We wish to use a $h(y) \simeq r > 0$. We use the Verhulst equation (oftentimes, "the logistic equation") such that:

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

Where $K = \frac{r}{a}$, and r is the intrinsic growth rate.

We seek simple, constant functions for y, such that $y = \phi_1 = 0$ and $y = \phi_2 = K$. We classify these solutions as *equilibrium solutions*. We further describe them by:

Stable equilibrium solution An equilibrium solution that is converged upon from both sides.

Semi-stable equilibrium solution An equilibrium solution that is converged upon from one, but not both sides.

Unstable equilibrium solution An equilibrium solution that is diverged upon from both sides.

A *phase line* is used as an accompanying context to the y-axis to, for each equilibria, show whether we are converging towards or away from that solution.

If we wish to solve the specific equation, we go as follows:

$$\frac{dy}{(1 - \frac{y}{K})y} = r\ dt$$

And perform a partial fraction expansion on the left-hand side as:

$$\left(\frac{1}{y} + \frac{\frac{1}{K}}{1 - \frac{y}{K}}\right) \, dy = r \; dt$$

Therefore,

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

Chapter 3

Second Order Differential Equations

A second-order, ordinary differential equation is of the form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Such a second-order differential equation is said to be *linear* if and only if f is of the form:

$$f = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

When in this case, we usually write the first equation as:

$$y'' + p(t)y' + q(t)y = g(t)$$

We focus on the analysis of *linear* second-order ODEs. In order to solve an Initial Value Problem, we require additional information about the initial condition of $y'(t_0)$. A second-order I.V.P. is given as:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

3.1 Homogeneity

A second-order ODE is called "homogeneous" when the term $g(t)$ (or, $G(t)$) is zero in all cases. A second-order ODE is called "non-homogeneous" when there exist values of t such that $g(t) \neq 0$.

3.2 Homogeneous Second-Order Differential Equations

Consider differential equations of the form

$$ay'' + by' + cy = 0$$

We will proceed to develop a method for solving such equations. We presume that solutions will be of the form $y = e^{rt}$, where r is a parameter yet to be determined. Therefore,

$$y' = re^{rt} \qquad \text{and} \qquad y'' = r^2 e^{rt}$$

We simplify to obtain:

$$(ar^2 + br + c)e^{rt} = 0$$

And, since $e^{rt} \neq 0$:

$$ar^2 + br + c = 0$$

Note that for any solutions $y_1(t)$, and $y_2(t)$, $y(t) = y_1(t) + y_2(t)$ is also a solution. Therefore, it follows also that:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \quad \Longleftrightarrow \quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Suppose that $y(t_0) = y_0$ and $y'(t_0) = y_0'$. We obtain the following:

$$y_0 = c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} \quad \text{and} \quad y_0' = c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0}$$

We solve each, to discover that:

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0} \quad \text{and} \quad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

3.3 Homogeneous Complex Second-Order Differential Equations

In the previous section, we considered roots only where the discriminant $(b^2 - 4ac)$ was positive. Consider instead the situation where $b^2 - 4ac < 0$. Thus, the roots r_1 and r_2 are the conjugates of complex numbers, and are denoted by:

$$r_1 = \lambda + i\mu \quad \text{and} \quad r_2 = \lambda - i\mu$$

Thus:

$$y_1 = e^{(\lambda + i\mu)t} \quad \text{and} \quad y_2 = e^{(\lambda - i\mu)t}$$

3.3.1 Euler's Formula

We define what it means to exponentiate by a complex function. Recall that the Taylor series for e^t (around $t = 0$) is given by:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad t \in \mathbb{R}$$

Assume that it is substitutable for t in the above equation. We separate the series expansion using the definition of exponentiating i. We therefore obtain:

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}}_{\text{Taylor Series, } \cos(t)} + i \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}}_{\text{Taylor Series, } \sin(t)} \\ &= \cos(t) + i \cdot \sin(t) \end{aligned}$$

Given that $e^{it} = \cos(t) + i \cdot \sin(t)$, we note that

$$e^{i\mu t} = \cos(\mu t) + i \cdot \sin(\mu t)$$

In general, we want for

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}$$

We therefore obtain that:

$$\begin{aligned} e^{(\lambda + i\mu)t} &= e^{\lambda t}(\cos(\mu t) + i \cdot \sin(\mu t)) \\ &= e^{\lambda t} \cos(\mu t) + i e^{\lambda t} \sin(\mu t) \end{aligned}$$

And that:

$$e^{(\lambda - i\mu)t} = e^{\lambda t} \cos(\mu t) - i e^{\lambda t} \sin(\mu t)$$

3.3.2 Simplifying when $b^2 - 4ac < 0$

When $b^2 - 4ac < 0$, the discriminant tells us that we have two complex solutions as above: $r = \lambda \pm i\mu$. We recall the judgement on y_1, y_2, to obtain that a solution is:

$$y = C_1 e^{(\lambda + i\mu)t} + C_2 e^{(\lambda - i\mu)t}$$

Recall that $e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos(\mu t) + i \cdot \sin(\mu t))$. Therefore, by some trigonometric identities, we obtain that:

$$\begin{aligned} y &= C_1 e^{(\lambda + i\mu)t} + C_2 e^{(\lambda - i\mu)t} \\ &= C_1 e^{(\lambda + i\mu)t} + C_2 e^{(\lambda - i\mu)t} \\ &= C_1 \left[e^{\lambda t} \cos(\mu t) + i e^{\lambda t} \sin(\mu t) \right] + C_2 \left[e^{\lambda t} \cos(\mu t) - i e^{\lambda t} \sin(\mu t) \right] \\ &= (C_1 + C_2) e^{\lambda t} \cos(\mu t) + (C_1 - C_2) i e^{\lambda t} \sin(\mu t) \\ &= a_1 e^{\lambda t} \cos(\mu t) + a_2 e^{\lambda t} \sin(\mu t) \end{aligned}$$

3.3.3 Solving the I.V.P

Now we wish to solve the Initial Value Problem for the above. Suppose that:

$$\begin{cases} y = a_1 e^{\lambda t}\cos(\mu t) + a_2 e^{\lambda t}\sin(\mu t) \\ y(0) = \alpha \\ y'(0) = \beta \end{cases}$$

Begin by differentiating y to obtain:

$$\begin{aligned} y' &= \frac{d}{dt}\left[a_1 e^{\lambda t}\cos(\mu t) + a_2 e^{\lambda t}\sin(\mu t)\right] \\ &= \frac{d}{dt}\left[e^{\lambda t}(a_1\cos(\mu t) + a_2\sin(\mu t))\right] \\ &= \lambda e^{\lambda t}(a_1\cos(\mu t) + a_2\sin(\mu t)) + e^{\lambda t}(-\mu a_1\sin(\mu t) + \mu a_2\cos(\mu t)) \\ &= \lambda y(t) + e^{\lambda t}(-\mu a_1\sin(\mu t) + \mu a_2\cos(\mu t)) \end{aligned}$$

Noting that we make the final substitution of $y(t)$ in the last line in order to make the numerical solution easier to manage. Often the initial value problem will be given in terms of $y(0)$, and $y'(0)$, which makes the $\sin(\mu t) \to 0$ and $\cos(\mu t) \to 1$.

A common example is shown below. We begin with solving for $y(0) = \alpha$:

$$\begin{aligned} y(0) = \alpha &= a_1 e^{\lambda \cdot 0}\cos(\mu \cdot 0) + a_2 e^{\lambda \cdot 0}\sin(\mu \cdot 0) \\ &= a_1 \end{aligned}$$

Now we solve for $y'(0) = \beta$:

$$\begin{aligned} y'(0) = \beta &= \lambda y(0) + e^{\lambda t}(-\mu a_1\sin(\mu t) + \mu a_2\cos(\mu t)) \\ &= \lambda\alpha + \mu a_2 \end{aligned}$$

3.4 Homogeneous Second-Order Differential Equations with Repeated Roots

Now we move our consideration to when the discriminant says that there are repeated, real roots, i.e., when $b^2 - 4ac = 0$.

We consider the general case of

$$ay'' + by' + cy = 0$$

We note that

$$r = r_1 = r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

and therefore that one such solution is given as

$$y_1(t) = e^{(-b/2a)t}$$

We now perform the d'Alembert procedure, and let

$$y(t) = v(t)y_1(t) = v(t)e^{(-b/2a)t}$$

We substitute our solution for $y(t)$ into the original differential equation. In order to do so, we must first obtain $y''(t)$ and $y(t)$. We go as follows:

$$y'(t) = v'(t)e^{(-b/2a)t} + \frac{-b}{2a}v(t)e^{(-b/2a)t}$$

$$y''(t) = v''(t)e^{(-b/2a)t} - \frac{b}{a}v'(t)e^{(-b/2a)t} + \left(\frac{b}{2a}\right)^2 v(t)e^{(-b/2a)t}$$

Then, we substitute into $ay'' + by' + cy = 0$ to obtain:

$$e^{(-b/2a)t}\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \left(\frac{b}{2a}\right)^2 v(t)\right] + b\left[v'(t) + \frac{-b}{2a}v(t)\right] + c \cdot v(t)\right\} = 0$$

Which we simplify to:

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0$$

And further until we note that:

$$v''(t) = 0 \qquad \text{and} \qquad v(t) = c_1 + c_2 t$$

Therefore y is solved by:

$$y = C_1 \cdot e^{(-b/2a)t} + C_2 t \cdot e^{(-b/2a)t}$$

3.4.1 Reduction of Order

Suppose that we have a solution, $y_1(t)$ for the following differential equation:

$$y'' + p(t)y' + q(t)y = 0$$

And we wish to obtain a second solution. Let:

$$y = v(t)y_1(t)$$

then:

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and:

$$\begin{aligned} y'' &= v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t) \\ &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{aligned}$$

We then substitute as above and obtain that:

$$y_1 \cdot v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$

therefore:

$$y_1'' + (2y_1' + py_1)v' = 0$$

Which is a first order linear differential equation of v', and we can solve it using our usual methods. Then, we can find $v' \rightarrow_{\int} v$.

3.5 Summary

When considering differential equations of the form:

$$ay'' + by' + cy = 0$$

We write the "characteristic equation" of the above as follows:

$$ar^2 + br + c = 0$$

Noting that r is solved by the quadratic formula as:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And we solve the differential equation using one of three methods based on the *discriminant* of the characteristic equation:

$b^2 - 4ac > 0$ There exist two real, distinct roots $r_1, r_2 \in \mathbb{R}$ where $r_1 \neq r_2$, such that an initial value problem can be solved by:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$b^2 - 4ac = 0$ There exists a single real, repeated root $r \in \mathbb{R}$, such that the initial value problem is solved by:

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

$b^2 - 4ac < 0$ There exist two complex, distinct roots $r_1, r_2 \in \mathbb{Z}$, where $r_1 \neq r_2$, such that an initial value problem can be solved by:

$$y(t) = e^{\lambda t} \left(C_1 \cos(\mu t) + C_2 \sin(\mu t) \right)$$

3.6 Method of Undetermined Coefficients

So far, we have solved equations of the form:

$$ay'' + by' + cy = 0$$

But what if we wanted to solve an equation of the form:

$$ay'' + by' + cy = g(t)$$

where $g(t) \neq 0$? To do this, we introduce and make use of the "Method of Undetermined Coefficients"[1].

[1] These notes are based on Loveless', which are found here: `https://sites.math.washington.edu/~aloveles/Math307Spring2016/m307Review3-5.pdf`.

We begin by solving a simpler differential equation, where $g(t) = 0$ using one of the methods above to obtain two independent solutions to the homogeneous equation, $y_1(t)$, and $y_2(t)$.

Then, we consider $g(t)$, and use the following table to find the likely form for a particular solution, $Y(t)$:

$g(t)$	e^{rt}	$\sin(\mu t)$ or $\cos(\mu t)$	C	t	t^2	t^3
$Y(t)$	Ae^{rt}	$A\cos(\mu t) + B\sin(\mu t)$	A	$At + B$	$At^2 + Bt + C$	$At^3 + Bt^2 + Ct + D$

Figure 3.1: Common particular solutions

Combining particular solutions If $g(t)$ is a sum or difference of any of the above types, then so is $Y(t)$. Further, if $g(t)$ is a product, then so is $Y(t)$, with the additional caveat that extra coefficients may be "factored" out.

Adjusting for homogeneous solutions If $Y(t)$ contains a *constant* multiple of either $y_1(t)$ or $y_2(t)$, then multiply by t (e.g., $Y(t) = t^n \cdot Y(t)$) until this is no longer the case.

Once we have an adjusted particular solution, we use it to find $Y'(t)$ and $Y''(t)$, and substitute these values into the original equation in order to find the constants A, B, and so on in Y.

Finally, the general solution is given as:

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + Y(t).$$

We can then use the initial conditions $y(0) = \alpha$ and $y'(0) = \beta$ in order to solve for C_1 and C_2.

3.7 Mechanical Vibrations

Consider a mass m hanging at rest on a prefect spring of original length l. When the mass m is attached, the spring elongates to L. In the static case, there are two forces at play:

$$F_g = -mg \qquad \text{and} \qquad F_s = -kL$$

F_g is the downwards pull of gravity on the mass-spring system, and F_s is a consequence of Hooke's Law, that the force on a perfect spring is proportional (with proportionality constant k) to the elongated distance, L.

Let $u(t)$ denote the displacement of the mass from the equilibrium position at time t, where $u(t)$ grows downward. Note that by Newton's Second Law:

$$mu''(t) = f(t)$$

We observe the following:

1. The weight $F_g = mg$ acts downward.
2. The spring force F_s is proportional to the total elongation $L+u(t)$, $F_s(t) = -k(L+u(t))$.
3. The damping force, F_d, *opposes* the force of the motion of the mass. We denote this as: $F_d = \gamma u'(t)$.
4. $F(t)$ is the applied external force.

We rewrite this all as:

$$\begin{aligned} mu''(t) &= mg + F_s(t) + F_d(t) + F(t) \\ &= mg - k(L + u(t)) - \gamma u'(t) + F(t) \end{aligned}$$

To obtain an initial value problem:

$$\begin{cases} mu''(t) + \gamma u'(t) + ku(t) = F(t) \\ u(0) = u_0 \\ u'(0) = v_0 \end{cases}$$

3.7.1 Undamped Free Vibrations ($\gamma = 0$)

If there is no external force and no damping, we note that $F(t) = \gamma = 0$. In this case, the motion of a spring is described by the following:

$$mu''(t) + ku = 0$$

Which is a constant coefficient 2nd-order ordinary differential equation with complex roots. The solution is given by:

$$u = C_1 \cos(\mu t) + C_2 \sin(\mu t)$$

This is sometimes rewritten as either of the following:

$$\begin{aligned} \ldots &= R\cos(\omega_0 t - \delta) \\ &= R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t) \end{aligned}$$

Note that the parameters of the above equation are given as follows:

$$A = R\cos(\delta) \qquad \text{and} \qquad B = R\sin(\delta)$$

Therefore:

$$R = \sqrt{A^2 + B^2} \qquad \text{and} \qquad \tan(\delta) = \frac{B}{A}$$

Note that the *frequency* of this system is given as:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

$\omega_0 = \sqrt{k/m}$ is often called the *natural frequency*, whereas R is referred to as the *amplitude*. The dimensionless parameter δ is the *phase*.

3.7.2 Damped Free Vibrations ($\gamma > 0$)

When we consider damping, the differential equation is given as:

$$mu'' + \gamma u' + ku = 0$$

The roots of the characteristic equation of the above are, unsurprisingly:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

We consider (again) three cases on the discriminant of the characteristic equation above:

$\gamma^2 - 4km > 0$ In this case we say that the system is *overdamped*. The differential equation is solved by:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Note that in this case, the $\lim_{t\to\infty} u(t) = 0$.

$\gamma^2 - 4km = 0$ In this case we say that the system is *critically damped*. The differential equation is solved by:

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

$\gamma^2 - 4km < 0$ In this case we note that system is *underdamped*, i.e., there are two complex roots, where:

$$\lambda = \frac{-\gamma}{2m} \qquad \text{and} \qquad \mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$$

(Often the solutions are written as $r = \lambda \pm i\omega_0$). Thus, the solution looks like:

$$y = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t)$$

Which we often write as:

$$y = Re^{\lambda t} cos(\mu t - \delta)$$

Where $R = \sqrt{C_1^2 + C_2^2}$, $C_1 = R\cos(\delta)$, and $C_2 = R\sin(\delta)$. $\delta = \tan(B/A)$ (as above). The graphical solution of this initial value problem is an exponentially decreasing sinusoidal curve converging towards 0. We define a few properties of this system:

Quasi-frequency is $\mu = \dfrac{4mk - \gamma^2}{2m}$ radians per second.

Quasi-period is $T = \dfrac{2\pi}{\mu}$ seconds/wave.

Amplitude is given as $Re^{\lambda t}$, where $\lim_{t\to\infty} Re^{\lambda t} = 0$.

When $\gamma = 0$, we say that the system is *undamped*, and the solution is given as:

$$y = C_1\cos(\mu t) + C_2\sin(\mu t)$$

And the graphical solution oscillates with no decay.

3.7.3 Kirchhoff's Law

In a RLC-circuit, we denote the voltage source, resistance, inductance, and capacitance as: V, R, L, and C, respectively. Denote q as the charge. Kirchhoff's Law says:

$$Lq'' + Rq' + \frac{q}{C} - V(t) = 0$$

3.8 Forced Mechanical Vibrations

We consider an analysis of *forced* mechanical vibrations, i.e., when $F(t) \neq 0$. To focus our analysis, we consider only the case where $F(t)$ takes the form of:

$$F(t) = R\cos(\omega t)$$

Thus, the differential equation involving the displacement function $u(t)$ at time t, is written as:

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

3.8.1 Undamped Forced Vibrations ($\gamma = 0$)

This case is modeled by:

$$mu'' + ku = F_0 \cos(\omega t)$$

The homogeneous and particular solution(s) have the form:

$$u_c(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \text{and} \qquad u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

Note that u_p may also be written as:

$$u_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$$

depending on whether or not $\omega = \omega_0$. For more, see a previous discussion on the Method of Undetermined Coefficients.

We observe the following two cases on ω and ω_0:

Case 1 ($\omega \neq \omega_0$)

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

Case 2 ($\omega = \omega_0$)

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

Note that the limit of $\lim_{t\to\infty} t\sin(\omega_0 t)$ does not exist, and the function is unbounded! This is is a demonstrated effect of *resonance*.

3.8.2 Damped Forced Vibrations ($\gamma > 0$)

The general solution to this situation is given as:

$$u(t) = \underbrace{C_1 u_1(t) + C_2 u_2(t)}_{\text{Homogegnous solution(s)}} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{\text{Particular solution}}$$

Note that the homogeneous solution "dies out" as $t \to \infty$. This solution allows us to meet initial conditions, but is eventually dominated by the particular solution, which we call the *steady state solution* or *forced response*.

It is often convenient to write the solution in *wave form*, as:

$$U(t) = R\cos(\omega t - \delta)$$

Where:

$$R = \frac{F_0}{\Delta} \qquad \cos(\delta) = \frac{m(\omega_0^2 - \omega^2)}{\Delta} \quad \text{and} \quad \sin(\delta) = \frac{\gamma\omega}{\Delta}$$

$$\omega_0 = \sqrt{k/m} \qquad\qquad \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Thus, the amplitude of the steady state resonance, R is given as:

$$R = \frac{F_0}{k}\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}\right]^{-1/2}$$

Note that the following two limits exist:

$$\lim_{\omega \to 0} R = \frac{F_0}{k} \qquad \text{and} \qquad \lim_{\omega \to \infty} R = 0$$

Chapter 4

Laplace Transformations

4.1 Introduction

4.1.1 Improper Integral

Recall that an improper integral is one whose upper- or lower-limit is equal to $\pm\infty$. We evaluate these integrals as follows:

$$\int_a^\infty f(t)dt = \lim_{A\to\infty}\int_a^A f(t)dt$$

We say that these integrals *converge* when their limit is defined, and that they *divere* otherwise.

4.1.2 Piecewise Continuity

We say that a function f is *piecewise continuous* if and only if on an interval f can be partitioned where in each interval, f is both continuous, and approaches a finite limit at the endpoints when approached from within the interval.

Theorem 1. If f is piecewise continuous for $t \geq a$, $|f(t)| \leq g(t)$ when $t \geq M$, and $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ converges.

4.1.3 Definition of the Laplace Transformation

The *Laplace Transformation* is defined as follows:

$$\mathcal{L}\left\{f(t)\right\} = \int_\alpha^\beta K(s,t)f(t)dt = \int_0^\infty e^{-st}f(t)dt$$

Where $K(s,t)$ is the *kernel* of the transformation, and the limits of integration are given.

Theorem 2. Suppose that f is piecewise continuous on the interval $[0, A]$, and $|f(t)| \leq Ke^{at}$ (where $K, a, M \in \mathbb{R}$ and $K, M > 0$). Then, the transformation $\mathcal{L}\left\{f(t)\right\}$ exists.

Finally, note that the Laplace Transformation is linear:

$$\begin{aligned}\mathcal{L}\left\{C_1 f_1(t) + C_2 f_2(t)\right\} &= \int_0^\infty e^{-st}\left[C_1 f_1(t) + C_2 f_2(t)\right] dt \\ &= C_1 \int_0^\infty e^{-st} f_1(t) dt + C_2 \int_0^\infty e^{-st} f_2(t) dt \\ &= C_1 \mathcal{L}\left\{f_1(t)\right\} + C_2 \mathcal{L}\left\{f_2(t)\right\}\end{aligned}$$

4.2 Solutions to Initial Value Problems

Theorem 3. Suppose that f is continuous and f' is piecewise continuous on $[0, A]$. Suppose that $\exists K, a, M.\ |f(t)| \leq Ke^{at}$ for $t \geq M$. Then:

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$$

Corollary 1. Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous, and that $f^{(n)}$ is piecewise continuous on $[0, A]$. If there exists K, a, M (as above), such that for *any* n, $|f^{(n)}(t)| \leq Ke^{at}$, then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists, and is given as:

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\left\{f(t)\right\} - \sum_{i=1}^{n} s^{n-i} f^{i-1}(0)$$

4.2.1 Homogeneous Differential Equations

Consider the general second-order linear initial value problem:

$$\begin{cases} ay'' + by' + cy = 0 \\ y(0) = \alpha \\ y'(0) = \beta \end{cases}$$

We solve this equation using the Laplace Transformation. We begin as follows by taking the Laplace Transformation of both sides, and applying the linearity of the Laplace Transformation:

$$y = \mathcal{L}^{-1}\left\{\frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}\right\}$$

For more, see: A.1.

4.2.2 Non-homogeneous Differential Equations

Similarly, for initial value problems of the form:

$$\begin{cases} ay'' + by' + cy = f(t),\ f(t) \neq 0 \\ y(0) = \alpha \\ y'(0) = \beta \end{cases}$$

We go as above:

$$\mathcal{L}\left\{y(t)\right\}(s) = \frac{\mathcal{L}\left\{F(t)\right\}(s) + (asy(0) + ay'(0) + by(0))}{as^2 + bs + c}$$

For more, see: A.2.

4.2.3 Elementary Laplace Transformations

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s},\ s > 0$
2.	e^{at}	$\frac{1}{s-a},\ s > a$
3.	$t^n,\ n > 0$	$\frac{n!}{s^{n+1}},\ s > 0$
4.	$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}},\ s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2},\ s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2},\ s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2},\ s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2},\ s > \lvert a \rvert$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\ s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$
11.	$t^n e^{at},\ n > 0$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s),\ s > 0$
14.	$e^c f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right),\ c > 0$
16.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Table 4.1: Elementary Laplace Transformations

4.3 Inverse Laplace Transformations

Given a function $F(s)$, we wish to find $f(t)$ such that[1]:

$$\mathcal{L}^{-1}\left\{F(s)\right\}(t) = \mathcal{L}\left\{f(t)\right\}(s) \qquad \Longleftrightarrow \qquad \mathcal{L}\left\{f(t)\right\}(s) = \mathcal{L}^{-1}\left\{F(s)\right\}(t)$$

Often, a Laplace Transformation comes out to be a quotient of polynomial terms, as:

$$R(s) = \frac{P(s)}{Q(s)}$$

Before taking any further steps, we apply the following:

1. If the power of the numerator P is greater than or equal to the power of Q, then use long-division to rewrite R.
2. Factor Q, and complete the square otherwise.
3. Write the linear decomposition:
   - For terms of the form $(x-a)$, rewrite as: $\dfrac{A}{x-a}$.
   - For terms of the form $(x-a)^n$, rewrite as: $\displaystyle\prod_{i=1}^{n} \frac{A_i}{(x-a)^i}$

     For example, $(x-a)^3 = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3}$.
   - For quadratic terms (e.g., those of the form x^2+a^2), decompose as: $\dfrac{Ax+B}{x^2+a^2}$.
   - For repeated quadratic terms, decompose as: $\displaystyle\prod_{i=1}^{n} \frac{A_i x + B_i}{(x^2+a^2)^i}$.
4. Finally, solve and write the partial fraction decomposition. Then, use the Inverse Laplace Transformation (see: values from the above table) in order to compute $y(t)$.

There are a few tricks to speeding this process up:

4.3.1 "Cover-up" method

For each term in the decomposition with an unknown numerator (e.g., A, B, and so on...), take the root of the denominator and evaluate R at those roots. This will take Q to 0, but ignore these terms. The "covered up" value of R at each root is the value of that root's numerator in the decomposition.

[1] This section of notes are based on Loveless', which are found here: `https://sites.math.washington.edu/~aloveles/Math307Spring2015/m307PartialFractions.pdf`.

Example 4.3.1. Consider the partial fraction decomposition:

$$\frac{x^2+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

We find each by evaluating the original goal at the roots for A, B, and C, which are 0, -1, and -2, respectively.

$$\left.\frac{x^2+2}{\cancel{x}(x+1)(x+2)}\right|_{x=0} = 1 \qquad \Longrightarrow A = 1$$

$$\left.\frac{x^2+2}{x\cancel{(x+1)}(x+2)}\right|_{x=-1} = -3 \quad \Longrightarrow B = -3$$

$$\left.\frac{x^2+2}{x(x+1)\cancel{(x+2)}}\right|_{x=-2} = 3 \qquad \Longrightarrow C = 3$$

Hence:

$$\frac{x^2+2}{x(x+1)(x+2)} = \frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2}$$

4.3.2 Enhanced "cover-up" method

For repeated terms (e.g., linear/n-ary terms with power greater than 1), the above method is insufficient. Proceed as far as possible with it against linear terms, and then cross multiply and "match coefficients" to produce a system of equations that solves for the remaining terms.

Example 4.3.2. Consider the partial fraction decomposition:

$$\frac{5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

We use the cover up method to find that $A = 5/9$, and $C = 5/3$. To find B, we begin as:

$$\frac{5}{(x+1)(x-2)^2} = \frac{5}{9(x+1)} + \frac{B}{x-2} + \frac{5}{3(x-2)^2}$$

Solving for B, we obtain:

$$5 = \frac{5}{9}(x-2)^2 + \frac{B(x+1)}{x-2} + \frac{5}{3}(x+1)$$

One coefficient involving B is on the term labelled x^2. We find therefore that $B = -5/9$, hence:

$$\frac{5}{(x+1)(x-2)^2} = \frac{5}{9(x+1)} - \frac{5}{9(x-2)} + \frac{5}{3(x-2)^2}$$

4.3.3 Complex "cover-up" method

When the denominator has complex roots, find them using the quadratic formula, and R in terms of this (e.g., x^2+1 goes to $(x+i)(x-i)$). Use the cover-up method with $\pm i$, and solve.

4.4 Step Functions

Define the *unit step function* or the *Heaviside function* to be:

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

This function has many important applications; we study three ways that this function composes other functions here:

1. Piecewise functions of many values: we can add many unit step functions together with different values of c, and interpret their solutions graphically as well as numerically.
2. Piecewise functions can "mute" a section of a function, by multiplying it to that function, for a particular c.

We demonstrate the Laplace Transformation of the unit step function as follows:

$$\begin{aligned} \mathcal{L}\left\{u_c(t)\right\} = \int_0^\infty e^{-st} u_c(t) &= \int_c^\infty e^{-st} dt \\ &= \frac{e^{-cs}}{s},\ s > 0 \end{aligned}$$

Theorem 4. If $F(s) = \mathcal{L}\left\{F(t)\right\}$ exists, then:

$$\mathcal{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs}\mathcal{L}\left\{f(t)\right\} = e^{-cs}F(t)$$

Proof.

$$\begin{aligned} \mathcal{L}\left\{u_c(t) f(t-c)\right\} &= \int_0^\infty e^{-st} u_c(t) f(t-c) dt \\ &= \int_c^\infty e^{-st} f(t-c) dt \end{aligned}$$

Let $\xi = t - c$, and $d\xi = dt$. Therefore:

$$\begin{aligned} \mathcal{L}\left\{u_c(t) f(t-c)\right\} = \int_0^\infty e^{-(\xi+c)s} f(\xi) d\xi &= e^{-cs} \int_0^\infty e^{-s\xi} f(\xi) d\xi \\ &= e^{-cs}\mathcal{L}\left\{f(t)\right\} \end{aligned}$$

□

Theorem 5. If $F(s) = \mathcal{L}\left\{f(t)\right\}$ exists, then:

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c)$$

Conversely, if $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$, then:

$$e^{ct} f(t) = \mathcal{L}^{-1}\left\{F(s-c)\right\}$$

Proof.

$$\begin{aligned} \mathcal{L}\left\{e^{ct} f(t)\right\} = \int_0^\infty e^{-st} e^{ct} f(t) dt &= \int_0^\infty e^{-(s-c)t} f(t) dt \\ &= F(s-c) \end{aligned}$$

□

4.5 Impulse Functions

We wish for formalisms describing an initial "impulse" to a system that has certain properties which we describe below. We begin by definition a function d_τ which is given as:

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & t \in (-\tau, \tau) \\ 0 & t \notin (-\tau, \tau) \end{cases}$$

Let I be defined as:

$$I(t) = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \int_{-\infty}^{\infty} d_\tau dt$$

Note that:

$$\lim_{\tau \to 0^+} d_\tau(t) = 0,\ t \neq 0 \qquad \text{and} \qquad \lim_{\tau \to 0^+} I(t) = 1$$

I is in fact called δ, or the *Dirac delta function*

4.6 The Convolution Integral

Suppose that we are given a functions f, g, and h in the time domain. Further suppose that $H(s) = F(s)G(s)$. It is reasonable to conclude that $h(t) = fg$, but this is not the case. Instead, h is the *convolution* of f and g.

Theorem 6. If $F(s) = \mathcal{L}\{f(t)\}(s)$, and $G(s) = \mathcal{L}\{g(t)\}(s)$, then:

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}(s)$$

where:

$$h(t) = \int_0^t f(t-\tau)g(\tau)\ d\tau = \int_0^t f(\tau)g(1-\tau)\ d\tau$$

Proof. We know that:

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\ d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\tau} g(\tau)\ d\tau$$

thus:

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\ d\xi \int_0^\infty e^{-s\tau} g(\tau)\ d\tau$$

Since $\xi \neq \tau$, we can write the above as an iterated integral:

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-s\xi} f(\xi)\ d\xi \right] d\tau \\ &= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\ d\xi \right] d\tau \end{aligned}$$

Let $\xi = t - \tau$ for fixed τ, so that $d\xi = d\tau$. We rewrite the above integral as:

$$F(s)G(s) = \int_0^\infty g(\tau) \left[\int_\tau^\infty e^{-st} f(t - \tau)\, dt \right] d\tau$$

By Fubini's Theorem, we can reverse the order of integration to obtain:

$$F(s)G(s) = \int_0^\infty e^{-st} \left[\int_0^t f(t - \tau) g(\tau)\, d\tau \right] dt$$

or

$$F(s)G(s) = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}\left\{ h(t) \right\}(s).$$

□

In the above, the convolution h is often written as:

$$h = (f * g)(t)$$

Note that the symbol $*$ is *not* the same as composition, which is denoted $\circ$.

4.6.1 Properties of $*$

The convolution integral is commutative, meaning that it can be written as the convolution of its first and second operand, or its second and first operand.

$$f * g = g * f$$

It is also distributive, meaning that it distributes like $+$, and so on.

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

$*$ is associative, which means that the "grouping" of terms under convolution does not matter.

$$f * (g * h) = (f * g) * h$$

$*$ also has a "zero", meaning that if you convolute any function with 0, or 0 with any function, the resoling convolution is itself 0.

$$f * 0 = 0 * f = 0$$

4.6.2 Solving Initial Value Problems with Convolution

Suppose that we are given a classic initial value problem of the form:

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(0) = y_0 \\ y'(0) = y'_0 \end{cases}$$

Using a result proven in A.2, we obtain that:

$$(as^2 + bs + c)\mathcal{L}\{y(t)\}(s) - (as + b)y_0 - ay_0' = \mathcal{L}\{g(t)\}(s)$$

and letting:

$$\Phi(s) = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c} \qquad \text{and} \qquad \Psi(s) = \frac{\mathcal{L}\{g(t)\}(s)}{as^2 + bs + c}$$

we can write that:

$$\mathcal{L}\{y(t)\}(s) = \Phi(s) + \Psi(s)$$

or:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\Phi(s)\}(t) + \mathcal{L}^{-1}\{\Psi(s)\}(t) \\ &= \phi(t) + \psi(t) \end{aligned}$$

Crucially, note the following:

$$\phi(t) : \begin{cases} ay'' + by' + cy = 0 \\ y(0) = y_0 \\ y'(0) = y_0' \end{cases} \qquad \text{and} \qquad \psi(t) : \begin{cases} ay'' + by' + cy = g(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Let $H(s)$ be given as follows:

$$H(s) = \mathcal{L}\{h(t)\}(s) = \frac{1}{as^2 + bs + c}$$

thus, we write:

$$\Psi(s) = H(s)G(s)$$

and call H the *transfer function*. Thus,

$$\psi(s) = \mathcal{L}^{-1}\{H(s)G(s)\}(t) = \int_0^t h(t - \tau)g(\tau)\, d\tau = (h * g)(t).$$

Where h is the *impulse response* of the system, and is solved by:

$$h : \begin{cases} ay'' + by' + cy = \delta(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Appendix A

A.1 Laplace Transform for Homogeneous I.V.P.'s

$$
\begin{aligned}
\mathcal{L}\left\{ay'' + by' + cy\right\} &= \mathcal{L}\left\{0\right\} \\
a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= 0 \\
a\left(s\mathcal{L}\left\{y'\right\} - y'(0)\right) + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= 0 \\
a\left(s\left(s\mathcal{L}\left\{y\right\} - y(0)\right) - y'(0)\right) + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= 0 \\
a\left(s\left(s\mathcal{L}\left\{y\right\} - y(0)\right) - y'(0)\right) + b\left(s\mathcal{L}\left\{y\right\} - y(0)\right) + c\mathcal{L}\left\{y\right\} &= 0 \\
as^2\mathcal{L}\left\{y\right\} - asy(0) - ay'(0) + b\left(s\mathcal{L}\left\{y\right\} - y(0)\right) + c\mathcal{L}\left\{y\right\} &= 0 \\
as^2\mathcal{L}\left\{y\right\} - asy(0) - ay'(0) + bs\mathcal{L}\left\{y\right\} - by(0) + c\mathcal{L}\left\{y\right\} &= 0 \\
\mathcal{L}\left\{y\right\}\left(as^2 + bs + c\right) - asy(0) - ay'(0) - by(0) &= 0 \\
asy(0) + ay'(0) + by(0) &= \mathcal{L}\left\{y\right\}\left(as^2 + bs + c\right) \\
\frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} &= \mathcal{L}\left\{y\right\} \\
\boxed{\mathcal{L}^{-1}\left\{\frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}\right\}} &= y
\end{aligned}
$$

A.2 Laplace Transform for Non-homogeneous I.V.P.'s

$$
\begin{aligned}
\mathcal{L}\left\{ay'' + by' + cy\right\} &= \mathcal{L}\left\{f(t)\right\} \\
a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
a\left(s\mathcal{L}\left\{y'\right\} - y'(0)\right) + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
a\left(s\left(s\mathcal{L}\left\{y\right\} - y(0)\right) - y'(0)\right) + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
a\left(s\left(s\mathcal{L}\left\{y\right\} - y(0)\right) - y'(0)\right) + b\left(s\mathcal{L}\left\{y\right\} - y(0)\right) + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
as^2\mathcal{L}\left\{y\right\} - asy(0) - ay'(0) + b\left(s\mathcal{L}\left\{y\right\} - y(0)\right) + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
as^2\mathcal{L}\left\{y\right\} - asy(0) - ay'(0) + bs\mathcal{L}\left\{y\right\} - by(0) + c\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{f(t)\right\} \\
\mathcal{L}\left\{y\right\}\left(as^2 + bs + c\right) - asy(0) - ay'(0) - by(0) &= \mathcal{L}\left\{f(t)\right\} \\
\mathcal{L}\left\{f(t)\right\} + \left(asy(0) + ay'(0) + by(0)\right) &= \mathcal{L}\left\{y\right\}\left(as^2 + bs + c\right) \\
\frac{\mathcal{L}\left\{f(t)\right\} + \left(asy(0) + ay'(0) + by(0)\right)}{as^2 + bs + c} &= \mathcal{L}\left\{y\right\} \\
\boxed{\mathcal{L}^{-1}\left\{\frac{\mathcal{L}\left\{f(t)\right\} + \left(asy(0) + ay'(0) + by(0)\right)}{as^2 + bs + c}\right\}} &= y
\end{aligned}
$$