MATH308 A, Eufemia, Final Exam

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Contents

Chapter 4

Systems of Linear Equations

4.1 Lines and Linear Equations

4.1.1 Linear Equations

A *linear equation* is an equation of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{4.1}$$

Solutions, Solution Sets

A *solution* to the above is an n-tuple (where n is the number of unknown variables in the equation) of the form $(s_1, s_2, \ldots, s_n)$.

A *solution set* for a linear equations is a set of all solutions to a single equation. In practice, this means:

$n = 2$ Geometric line of solutions in the xy-plane.

$n = 3$ Geometric lane in $\mathbb{R}^3$-space.

$n \geq 4$ Hyperplane.

4.1.2 Systems of Equations

Systems of equations are written as a finite set of linear equations. They are written with their variables aligned, as follows:

$$\begin{aligned} -3x_1 + 5x_2 - x_3 &= 4 \\ - x_2 - 9x_3 &= -4 \\ 6x_1 - 4x_2 &= 11 \\ -5x_1 - 9x_2 &= 0 \end{aligned} \tag{4.2}$$

4.1.3 Systems of Linear Equations

A system of linear equations is a collection of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \quad + \cdots + a_{1n}x_n \quad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \quad + \cdots + a_{2n}x_n \quad &= b_2 \\ \vdots \qquad\qquad\qquad\qquad &\vdots \\ a_{n1}x_1 + a_{n2}x_2 \quad + \cdots + a_{mn}x_n \quad &= b_n \end{aligned} \tag{4.3}$$

Coefficients on the a-terms are read as "a-three-two" for the coefficient a_{32}.

Systems can have m equations with n unknowns. m can be greater than, less than, or equal to n. A solution to a linear system is an n-tuple as above $((s_1, s_2, \ldots, s_n))$ that satisfies every equation in the system. The collection of all solutions for a system of equations is called a solution set.

Consistency

Were a linear system to have at least one solution, than we say that that system is *consistent.* If it has no solutions, than the system is *inconsistent.*

Free Parameters

When describing a solution set, denote free parameters as s_n, meaning that they can be any real number. This notation constitutes the *general solution*, since it gives all solutions in the set.

4.1.4 Finding Solutions

Triangular Systems

Consider the following triangular system of equations:

$$\begin{aligned} x_1 - 2x_2 - 5x_3 + 3x_4 &= 2 \\ x_2 + 3x_3 - 4x_4 &= 7 \\ x_3 + 2x_4 &= -4 \\ x_4 &= 5 \end{aligned} \tag{4.4}$$

A technique used to solve this is known as "back-substitution". Working from the bottom up, we can see that x_4 *must* be equal to 5, which allows us to then substitute into the equation directly above it. This allows us to obtain the solution for x_3, and so on until no more free variables remain.

Leading variables are variables that lead at least one equation in the system. In the above example the leading variables are composed from the set: $\{x_1, x_2, x_3, x_4\}$.

In general, a system is in triangular form (and is therefore a triangular system) if it is in the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \quad a_{13}x_3 + \cdots + \quad a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \quad a_{23}x_3 + \cdots + \quad a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \quad a_{33}x_3 + \cdots + \quad a_{3n}x_n &= b_3 \\ \ddots \qquad \vdots &= \vdots \\ a_{nn}x_n &= b_n \end{aligned} \tag{4.5}$$

Triangular systems exhibit the following properties:

1. Every variable is the leading variable of exactly one equation in the system.
2. There are the same number of variables as there are equations ($n = m$).
3. There is exactly one solution to the system.

Echelon Systems

A system is in echelon form (and is therefore an echelon system) if:

1. Every variable is the leading variable of *at most* one equation.

2. There are none, one, or infinitely many solutions to the system.

In general, to find the solution set for an echelon system, first set each free variable to a new free parameter, and then solve the system using back-substitution.

4.2 Linear Systems and Matrices

Linear systems often do not come in echelon form. Here is presented a discussion on how to transform any system of linear equations into an equivalent system in echelon form.

4.2.1 Elementary Transformations

Two linear systems are equivalent if and only if they produce the same solution set. Linear systems can be transformed into different, but equivalent linear systems by applying the following transformations.

1. Interchange the position of two equations.
2. Multiply (both sides of) an equation by a non-zero constant.
3. Add a multiple of one equation to another.

4.2.2 Augmented Matrix

When applying a set of Elementary Transformations, it is useful to represent the linear system as an augmented matrix. The leftmost elements represent the coefficients of each variable in the system, and the rightmost elements represent the constants b_1, b_2, and etc.

The following two systems are equivalent representations of one another:

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= -1 \\ 2x_1 - 5x_2 - x_3 &= 2 \\ -4x_1 + 13x_2 - 12x_3 &= 11 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & -1 & 2 \\ -4 & 13 & -12 & 11 \end{array}\right] \tag{4.6}$$

4.2.3 Echelon Form

A matrix is in echelon form if:

1. There are an increasing number of leading zeros as each row of the matrix is descended.

2. All rows that contain exclusively zeros are at the bottom of the matrix.

Here is an example of a matrix in echelon form:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 11 \end{array}\right] \tag{4.7}$$

(This matrix has no solutions, since $0 \neq 11$.)

4.2.4 Gaussian Elimination

Here follows an algorithm for transforming that matrix into echelon form:

1. Beginning at Row 1, identify the first non-zero element (from left to right), which we will denote the position of as the "pivot position".

2. "Eliminate" the positions below that column by transforming them to be zeros by Elementary Operations (2) and (3).

3. Move to the bottom row, and repeat step (1), until the matrix is in echelon form.

4.2.5 Gauss-Jordan Elimination

Here follows an algorithm for row-reducing a matrix into row-reduced echelon form:

1. Multiply each row by the inverse of its pivot to make each pivot element equal to 1.

2. Use row operations to introduce zero elements above the pivot position, working in a similar fashion to Gaussian elimination, but selecting pivot elements from right to left.

4.2.6 Row-Reduced Echelon Form

A matrix is in Row-Reduced Echelon Form if:

1. It is in echelon form.
2. All pivot positions contain a 1.
3. The only non-zero element in a pivot column is the pivot itself.

Here is an example of a matrix in row-reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 11 \end{array}\right] \tag{4.8}$$

(Again, this matrix has no solutions since $0 \neq 11$.)

4.2.7 Homogeny

A linear equation is homogeneous if it has the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \tag{4.9}$$

A system of linear equations is homogeneous if it has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \quad &+ \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 \quad &+ \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \quad & \qquad\quad \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 \quad &+ \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{4.10}$$

All systems of homogeneous linear equations are consistent because there exists a trivial solution:

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0. \tag{4.11}$$

Non-trivial solutions follow by using Gaussian and Gauss-Jordan elimination techniques.

4.2.8 Proof of Solution Cardinality

Theorem 1. *A system of linear equations has either (1) one solution, (2) no solutions, or (3) infinitely many solutions.*

Proof. For any linear system, the Gaussian transformation of the augmented matrix can lead to one of three cases:

1. The transformed system can have an equation of the form $0 = c$, for $c \neq 0$. In this case, there will be no solutions.

2. The transformed system has no free variables, and therefore has only one solution.

3. The transformed system has one or more free variables, and thus must have infinitely many solutions.

Homogeneous systems are even simpler, since the first case (an equation of the form $0 = c$, for $c \neq 0$) cannot occur, and the case outcome must be one of (2), (3). □

Chapter 5

Euclidean Space

5.1 Vectors

A vector is an ordered list of real numbers, expressed as:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{5.1}$$

and a Euclidean Space in n dimensions is the set $\mathbb{R}^n$ composed of all vectors $\mathbf{u}_i$ that contain n entries.

5.1.1 Operations on Vectors

Equality two vectors in $\mathbb{R}^n$ are equal if and only if $\forall i \in n \mid \mathbf{u}_i = \mathbf{v}_i$.

Addition two vectors may be added together as is follows:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \tag{5.2}$$

Scalar Multiplication A vector may be multiplied by a scalar c as follows:

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c * u_1 \\ c * u_2 \\ \vdots \\ c * u_n \end{bmatrix} \tag{5.3}$$

5.1.2 Algebraic Properties of Vectors

This section details the standard Algebraic Properties of Vectors, which may be applied in computation axiomatically:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{c}) = (\mathbf{u} + \mathbf{v}) + \mathbf{c}$
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\mathbf{u} + -\mathbf{u} = 0$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $(cd)\mathbf{u} = c(d\mathbf{u})$
- $1\mathbf{u} = \mathbf{u}$

5.1.3 Linear Combinations & Systems of Equations

If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are vectors, and $c_1, c_2, \ldots, c_n$ are scalars, then:

$$c_1u_1 + c_2u_2 + \ldots + c_nu_n \tag{5.4}$$

is the linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Note that the coefficients c_n (etc.) may be zero, or negative.

Systems of Equations

To express a system of equations, let the coefficients be $c_1 = x_1, c_2 = x_2, \ldots, c_n = x_n$, and so on:

$$x_1 \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \\ 11 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 5 & -1 & 6 & 12 \\ -3 & 2 & 1 & -4 \\ 7 & 5 & 0 & 11 \end{array}\right] \tag{5.5}$$

The solution to a system of equations is a column vector in the same Euclidean Space as the system is defined within.

5.1.4 Geometry of Vectors

Tip-to-Tail Rule If two vectors are arranged so that the end of one vector is in the same position as the start of the other vector, then the vector summation of the two is drawn from the start of the first vector to the end of the second.

Parallelogram Rule If two vectors are arranged such that the two vectors begin at the same place, then a parallelogram is drawn that contains the two vectors as significant edges, and the vector summation is drawn from the common base to the far corner of the parallelogram.

5.2 Span

Here begins a formal definition of the term *span*:

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a set of vectors in $\mathbb{R}^n$. The span of this set is noted as span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, and is defined as the set of all linear combinations of:

$$\{x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_n\mathbf{u}_n \mid x_i \in \mathbf{R}\}. \tag{5.6}$$

such that:

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} = \{x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_n\mathbf{u}_n \mid x_i \in \mathbf{R}\}. \tag{5.7}$$

5.2.1 Span in $\mathbf{R}^3$

The span of two vectors in $\mathbf{R}^3$ has exactly three possibilities:

1. Both vectors are $\mathbf{0}$, and thus cannot span any portion of $\mathbb{R}^3$.
2. Both vectors are scalar multiples of one another, and thus the span is a line in $\mathbb{R}^3$.
3. Both vectors satisfy neither of the above conditions, and thus the span is a plane for which the normal vector is the cross product of the two vectors, $\mathbf{u}_1 \times \mathbf{u}_2$.

If a third vector is introduced, three possibilities remain:

1. The third vector is $\mathbf{0}$, and thus the span is unchanged.
2. The third vector is in the plane spanned by the above, and thus the span is unchanged.
3. The third vector is *not* in the plane, and thus the span is all of $\mathbb{R}^3$.

5.2.2 Span Inclusion Proof

Theorem 2. *Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. $\mathbf{v}$ is an element of* $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ *if and only if the following equation has a solution:*

$$\left[\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m & \mathbf{v} \end{array}\right] \tag{5.8}$$

Proof. By the definition of solving Echelon Form systems in an Augmented Matrix, there will exist an $x_1, x_2, \ldots, x_m$, such that $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots + x_m\mathbf{u}_m = \mathbf{v}$. □

5.2.3 Span of a Linear Combination

Theorem 3. *Let $\mathbf{u} \in \operatorname{span}\{\mathbf{u}_1 \ldots \mathbf{u}_m\}$. Then:*

$$\operatorname{span}\{\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_m\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}. \tag{5.9}$$

5.2.4 Spanning $\mathbb{R}^n$

Theorem 4. *Suppose that* $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^n$*, and let:*

$$A = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m] \sim B \tag{5.10}$$

Then $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *is* $\mathbb{R}^n$ *when* B *is in Echelon Form and has a pivot position in every row.*

Proof. By case analysis on B:

B **does have a pivot position in every row** An equivalent echelon form transformation will yield a Echelon Form system of A such that the rightmost column has no pivot positions, and thus has a free variable and infinitely many solutions. Thus, it will be inconsistent and span all of $\mathbb{R}^n$.

B **does not have a pivot position in every row** If B does not have a pivot position in every row, then consider the example of the last row of B being the 0 row vector. $\mathbf{v} = (\ldots, c)$ (such that $c \neq 0$) is automatically not included in the span since that would yield a false equality statement, and thus since $\mathbf{v} \in \mathbb{R}^n$, the span of the set does not cover $\mathbb{R}^n$.

□

5.2.5 Solving $Ax = B$

Assume that A is the coefficient matrix, x is the column vector of variables in the system, and B is the solution matrix, then:

$$\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} a + 2b \\ 3a + 4b \end{bmatrix} \tag{5.11}$$

5.3 Linear Independence

Here follows two key ideas of linear independence, and their relation to the span operation:

Linear dependence The description of a set of vectors spanning some portion of Euclidean space can be reduced and still span the same subspace.

Linear independence The description of a set of vectors such that reducing the set size would also reduce the span of the subspace.

In other terms, let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. If the only solution to the vector equation:

$$x_1\mathbf{u_1} + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0} \tag{5.12}$$

is trivial, then the set is linearly independent. Otherwise, if a nontrivial solution exists, then the set is linearly dependent.

5.3.1 Related Theorems

Theorem 5. *Let $\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. This set is linearly dependent.*

Proof. By contradiction. Suppose that $x_1 = x_2 = \cdots = x_m = 0$. Let $x_0 = c \in \mathbb{R}$. □

Theorem 6. *Let $\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. If $n < m$, the set is linearly dependent.*

Proof. Since there are more parameters than equations, the solution space is inconsistent, and thus has infinitely many solutions, therefore, non-trivial solutions exist and the system is linearly dependent. □

Theorem 7. *Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. The set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.*

Theorem 8. *Let $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Suppose:*

$$A = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdots \\ \mathbf{u} \end{bmatrix} \sim B \tag{5.13}$$

where B is in echelon form. Then:

1. $\operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} = \mathbb{R}^n$ *when B has pivot positions in every row.*

2. *$\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly independent when B has a pivot position in every column.*

Proof. 1. Prove (1) with Theorem 2.8 (Holt).

2. Prove (2) by the following:

   - If B has a pivot position in every column, then the homogeneous system has a unique solution since there are no free variables. The solution must be trivial, therefore the system is linearly independent.
   - Otherwise, were B to have a pivot position in every row, then the homogeneous system would have parameters, and therefore an infinite count of solutions, thus the system would be linearly dependent.

□

5.3.2 Homogeneous Systems

A system is homogeneous if and only if it is expressible given the following form:

$$A\mathbf{x} = \mathbf{b}. \tag{5.14}$$

such that $\mathbf{b} = \mathbf{0}$.

Theorem 9. *The set of vectors $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m]$ is linearly independent if the homogeneous system has only the trivial solution:*

$$A\mathbf{x} = \mathbf{0}. \tag{5.15}$$

Proof. By an earlier theorem (trivially). □

Further, if $\mathbf{b} \neq \mathbf{0}$, then the system is non homogeneous and the associated homogeneous system is $A\mathbf{x} = \mathbf{0} = \mathbf{b}'$.

5.3.3 Particular Solution, General

Let $\mathbf{x}_p$ be the particular solution to a system. All solutions to the system $A\mathbf{x} = \mathbf{b}$, therefore, have the form $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$.

5.3.4 Unifying Theorem

Theorem 10. *Suppose* $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*, and is a set of vectors in* $\mathbb{R}^n$*. Let also* $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$*. The following statements are therefore equivalent:*

- $\operatorname{span} S = \mathbb{R}^n$.
- S *is linearly independent.*
- $A\mathbf{x} = \mathbf{b}$ *has a unique solution for all* b *in* $\mathbb{R}^n$*.*

5.3.5 Remarks

The following claims are equivalent:

- The set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent.
- The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}2 + \ldots + x_m\mathbf{a}_m = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
- The above, in equivalent notations.

Chapter 6

Matrices

6.1 Linear Transformations

Here begins a definition of the term Linear Transformation:

$$\{T : \mathbb{R}^m \to \mathbb{R}^n \mid \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^m, \forall r \in \mathbb{R} \to T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \wedge \\ \to T(r\mathbf{u}) = rT(\mathbf{u})\}.$$

or, stated in a simpler fashion:

$$T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v}). \tag{6.1}$$

6.1.1 Dimensionality

Dimensions If A is a matrix that has dimensions $n \times m$, then it is a $n \times m$-matrix.

Square Matrix If A is a matrix that has dimensions $n \times m$ (s.t., $n = m$), then A is a square matrix.

Related Theorems

Theorem 11. *If A is a $n \times m$ matrix, and $T(\mathbf{x}) = A\mathbf{x}$, then $T : \mathbb{R}^m \to \mathbb{R}^n$.*

Proof. Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, we have:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= A(\mathbf{u}+\mathbf{v}) \\ &= A\mathbf{u}+A\mathbf{v} \\ &= T(\mathbf{u})+T(\mathbf{v}). \end{aligned}$$

□

Theorem 12. *Let* $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m]$ *be an* $n \times m$ *matrix. Let* $T : \mathbb{R}^m \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$*. Then:*

1. $\mathbf{w} \in \text{range}(T)$ *iff* $A\mathbf{x} = \mathbf{w}$ *is a consistent system.*
2. $\text{range}(T) = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

Proof. A vector $\mathbf{w}$ is in range of T if there is a vector $T(\mathbf{w}') = \mathbf{w}$. By simplification, $T(\mathbf{w}') = A\mathbf{w}' = \mathbf{w}$, which requires that the system be consistent.

From 2.11, it follows that $A\mathbf{x} = \mathbf{w}$ is consistent when $\mathbf{w}$ is in the span of the columns of A. Trivially, $\text{range}(T) = \text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. □

6.1.2 Classifications of Linear Transformations

One-to-one, injective Every input vector maps to every output vector at most once (i.e., once, or not at all).

Onto, surjective Every output vector is mapped to by at least one input vector (i.e., one, n, or ∞).

6.1.3 Classification Theorems

Theorem 13. *Let* $T : \mathbb{R}^m \to \mathbb{R}^n$ *be one-to-one if and only if the only solution to* $T(\mathbf{x}) = \mathbf{0}$ *is trivial.*

Proof. Suppose that T is a linear transformation as defined. Therefore, it must have at most one solution to $T(\mathbf{x}) = \mathbf{0}$. Since it is a linear transformation, it follows that $T(\mathbf{0}) = \mathbf{0}$, and therefore the only solution is the trivial one. □

Theorem 14. *Let* A *be an* $n \times m$*-matrix, and define* $T : \mathbb{R}^m \to \mathbb{R}^n = \mathbf{A}x$*. Then:*

1. *T is **one-to-one** if and only if the columns of A are linearly independent.*

2. *If $A \sim B$ and B is in echelon form, then T is one-to-one if and only if B has a pivot position in every column.*

3. *If $n < m$, then T is not one-to-one.*

Proof. 1. T is **injective** if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. By an earlier theorem, this is true if and only if the columns are linearly independent.

2. This follows from the above and an earlier theorem.

3. The columns must necessarily be linearly dependent, thus A is non-surjective.

□

Theorem 15. *Let A be an $n \times m$-matrix and let T be a linear transformation such that $T : \mathbb{R}^m \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. Then:*

1. *T is onto if and only if the columns of A span $\mathbb{R}^n$.*

2. *If $A \sim B$ and B is in Echelon form, then T is onto if and only if B has a pivot position in every row.*

3. *If $n > m$, then T is not onto.*

Proof. 1. If T is onto then $\text{range}(T) = \mathbb{R}^n$, and therefore the columns must span the entire space.

2. From an earlier theorem, and (1).

3. If $n > m$, then the columns cannot span the co-domain, and thus the function is not surjective.

□

Theorem 16. *Let $T : \mathbb{R}^m \to \mathbb{R}^n$. T is a linear transformation if and only if $\exists A \to T(x) = A(x)$*

6.1.4 The Unifying Theorem, Part II

Theorem 17. *Let* $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *be a set of* n *vectors in* $\mathbb{R}^n$. *Let* $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, *and let* $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. *The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. S *is linearly independent.*
3. $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\forall \mathbf{b} \in \mathbb{R}^n$.
4. T *is one-to-one, or injective.*
5. T *is onto, or surjective.*

Proof. From the original statement of the Unifying Theorem, we know (1), (2), and (3). It follows that (1) and (4) are equivalent, and that (2) and (5) are equivalent. □

6.2 Matrix Algebra

6.2.1 Basic Algebraic Operations

Here we will define an algebra for working with matrices:

Addition & Subtraction Two matrices of the same dimension can be added and/or subtracted to one another by adding or subtracting each element to form a new matrix of the same dimension.

Scalar Multiplication A matrix may be multiplied by a scalar such that all elements are also multiplied by that scalar.

6.2.2 Properties of Algebra

Theorem 18. *Let* s *and* t *be scalars, and* A, B, C *be matrices of dimension* $n \times m$. *The following properties hold:*

1. $A + B = B + A$
2. $s(A + B) = sA + sB$

3. $(s+t)A = sA + tA$
4. $(A+B)+C = A+(B+C)$
5. $(st)A = s(tA)$
6. $A + \mathbf{0} = A$

6.2.3 Matrix Multiplication

Consider $B = [\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}]$ (where each $\mathbf{b_i}$ has k components), over a $n \times k$ matrix A. The result will be a $n \times m$ matrix, as follows:

$$AB = [A\mathbf{b_1}, A\mathbf{b_2}, \ldots, A\mathbf{b_m}]. \tag{6.2}$$

6.2.4 Identity Matrix

Additive identity Given $A_{n\times m}$ matrix, the additive identity is a matrix of the same dimension with 0-valued elements.

Left multiplicative identity Given $A_{n\times m}$ matrix, the left multiplicative identity is a square matrix of dimension n with 1s spanning the diagonal.

Right multiplicative identity Given $A_{n\times m}$ matrix, the right multiplicative identity is a square matrix of dimension m, with 1s spanning the diagonal.

6.2.5 Properties of Matrix Algebra

Theorem 19. *Consider a scalar $s \in \mathbb{R}$, and three matrices, A, B, C. The following holds where the operations are defined:*

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $s(AB) = (sA)B = A(sB)$

5. $AI = A$
6. $IA = A$

Note: I (as above) denotes an identity matrix of the proper dimensions.

Note the following facts do *not* necessarily follow from the above theorem:

1. $AB \neq BA$.
2. $AB = 0$ does *not* imply that A or B are equal to 0.
3. $AC = BC$ does not imply that $A = B$ or $C = 0$.

6.2.6 Transposition of a Matrix

Denote the transposition operation over a matrix as interchanging the rows with the columns. Therefore:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix} \tag{6.3}$$

Additionally, the following theorem is true:

Theorem 20. *Let A and B be $n \times m$ matrices. Let C be an $m \times k$ matrix, and $s \in \mathbb{R}$.*

1. $(A + B)^T = A^T + B^T$.
2. $(sA)^T = s(A^T)$.
3. $(AC)^T = C^T A^T$.

6.2.7 Compositions of Linear Transformations

Theorem 21. *Let $S : \mathbb{R}^m \to \mathbb{R}^k$, $T : \mathbb{R}^k \to \mathbb{R}^n$. By definition, there are matrices A_S, A_T, such that $S(\mathbf{x}) = A_S\mathbf{x}$, and $T(\mathbf{x}) = A_T\mathbf{x}$. The combined linear transformation $W : \mathbb{R}^m \to \mathbb{R}^n$ is expressed as:*

$$\begin{aligned} W(\mathbf{x}) &= T(S(\mathbf{x})) \\ &= A_S A_T \mathbf{x} \end{aligned}$$

6.2.8 Powers of a Matrix

Define the n-th power of a matrix A as follows:

$$A^n = \prod_{k=1}^{n} A. \tag{6.4}$$

Powers of a Diagonal Matrix

Let A be a square matrix of dimension m. Therefore, the exponentiation of a square diagonal matrix may be simplified as:

$$\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{mm} \end{bmatrix}^n = \begin{bmatrix} {a_{11}}^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {a_{mm}}^n \end{bmatrix} \tag{6.5}$$

6.2.9 Triangular Matrices

Upper-triangular Matrix A square matrix of dimension $m \times m$ where the lower-left half (across a diagonal) is zero.

Lower-triangular Matrix A square matrix of dimension $m \times m$ where the upper-right half (across a diagonal) is zero.

Triangular Matrix A matrix is Triangular if it is either Upper-triangular or Lower-triangular.

Theorem 22. *Let A be an $n \times m$ upper (or lower) triangular matrix. For all $k \in \mathbb{Q} \mid k \geq 1$, A^k is also an upper (or lower) triangular matrix.*

6.2.10 Elementary Matrices

Consider a set of Elementary Row Operations (E.R.O.'s) performed on a given matrix A to arrive at $\tilde{A}$ ($A \mapsto_Q \tilde{A}$). Then $I \mapsto_Q \tilde{I}$, and $A\tilde{I} = \tilde{A}$.

6.3 Inverses

6.3.1 Inverse Transformations

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is invertible if it is bijective (injective and surjective). If a matrix is invertible, then T^{-1} exists such that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^m$, $T(T^{-1}(\mathbf{x})) = \mathbf{x}$, and $T^{-1}(T(\mathbf{x})) = \mathbf{x}$.

Note that the following, therefore, must be true:

1. n must be strictly equal to m.
2. If T is invertible, then T^{-1} is also a linear transformation.

A matrix A is invertible when the following conditions are met:

1. A must be square (have dimension $n \times n$.)
2. There is a square matrix B (of the same dimension) such that $AB = I_n$.

Theorem 23. *Suppose that A is an invertible matrix, such that there exists a B where $AB = BA = I$. B is unique.*

6.3.2 Inverse Matrices

If A is an $n \times n$ matrix that is invertible, then A^{-1} is the inverse of A.

Non-singular The term used to describe a matrix A that is invertible.

Singular The term used to describe a matrix A that is not invertible.

Theorem 24. *Suppose that A, B are invertible $n \times n$ matrices, and C, D are non-invertible $n \times m$ matrices. Then:*

1. *A^{-1} is invertible such that $(A^{-1})^{-1} = A$.*
2. *AB is invertible such that $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *If $AC = AD$ then $C = D$.*
4. *If $AC = 0$, then $C = 0$.*

6.3.3 Finding Inverses

Apply an arbitrary sequence of Elementary Row Operations Q such that:

$$\left[A \mid I_n\right] \mapsto_Q \left[I_n \mid A^{-1}\right] \tag{6.6}$$

If after applying Q, the matrix is in row-reduced echelon form, then the matrix on the right will be A^{-1}. If the matrix on the left cannot be row-reduced, then A is non-invertible (read: A is singular).

Over 2×2 matrices

All 2×2 matrices are invertible, and are calculable by the following "quick formula". Suppose that:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6.7}$$

then:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{6.8}$$

6.3.4 The Unifying Theorem, Part III

Theorem 25. *Let* $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *be a set of* n *vectors in* $\mathbb{R}^n$*. Let* $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$*, and let* $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$*. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. S *is linearly independent.*
3. $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\forall \mathbf{b} \in \mathbb{R}^n$.
4. T *is one-to-one, or injective.*
5. T *is onto, or surjective.*
6. A *is invertible.*

Chapter 7

Subspaces

7.1 Introduction to Subspaces

A subset S of $\mathbb{R}^n$ is a subspace if it satisfies the following three conditions:

1. $\mathbf{0} \in S$.
2. $\mathbf{u}, \mathbf{v} \in S \to \mathbf{u} + \mathbf{v} \in S$. i.e., S is closed over scalar addition.
3. $r \in \mathbb{R}, \mathbf{u} \in S \to r\mathbf{u} \in S$. i.e., S is closed over scalar multiplication.

7.1.1 Classification of Subspaces

To classify whether a set S is a subspace $S \subseteq \mathbb{R}^n$, first begin by checking whether $\mathbf{0} \in S$. Then check if two "obvious" vectors included in S are also found in the sum of their vectors. If the set S is closed over addition, check again for scalar multiples. If no violations are found, then S is a subspace of $\mathbb{R}^n$.

Theorem 26. *Let* $S = \operatorname{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$. *Then* S *is a subspace of* $\mathbb{R}^n$.

7.1.2 null-spaces

Let A be an $n \times m$ matrix. The set of solutions to $A\mathbf{x} = \mathbf{0}$ is a null space and is denoted by $\operatorname{null} A$.

Theorem 27. *Let* A *be an* $n \times m$ *matrix.* $\operatorname{null}(A) = S$, S *is a subspace of* $\mathbb{R}^m$.

7.1.3 ker, & range of T

image(T) $\{\mathbf{y} \mid \exists \mathbf{x} : T(\mathbf{x}) = \mathbf{y}\}$.

ker(T) $\{\mathbf{x} \mid T(\mathbf{x}) = \mathbf{0}\}$.

In other terms, the image (image) operation on a linear transformation T is everything in the output domain of T for which there exists an input such that T will transform it to the output.

Likewise, the kernel of T (denoted ker) is the set of all input vectors to T that are transformed under T to $\mathbf{0}$.

Theorem 28. *Let* $T : \mathbb{R}^m \to \mathbb{R}^n$*.*

1. ker(T) *is a subspace of the domain in* $\mathbb{R}^m$*.*
2. image(T) *is a subspace of the subset of the codomain in* $\mathbb{R}^n$*.*

Theorem 29. *Let* $T : \mathbb{R}^m \to \mathbb{R}^n$*. T is one-to-one if and only if* $\ker(T) = \{\mathbf{0}\}$*.*

Therefore, we may amend The Big Theorem to include the above theorem:

Theorem 30. *Let* $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *be a set of n vectors in* $\mathbb{R}^n$*. Let* $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$*, and let* $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$*. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$*.*
2. *S is linearly independent.*
3. $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\forall \mathbf{b} \in \mathbb{R}^n$*.*
4. *T is one-to-one, or injective.*
5. *T is onto, or surjective.*
6. *A is invertible.*
7. ker(T) *is* $\{\mathbf{0}\}$*.*

7.2 Basis and Dimension

Denote $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ as the basis for a subspace S if the following are met:

1. $\operatorname{span} \mathcal{B} = S$.
2. $\mathcal{B}$ is linearly independent.

Theorem 31. *Let $\mathcal{B} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a basis for a subspace S. Then:*

$$\forall \mathbf{s} \in S \mid \exists s_1, \ldots, s_m : \mathbf{s} = \sum_{i=0} s_i \mathbf{u_i}. \tag{7.1}$$

7.2.1 Finding a Basis

To find a basis for a set of vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$, apply Gauss elimination on a matrix A where each *row* is $\mathbf{u_i}$, and the non-zero rows are the basis $\mathcal{B}$ of S.

7.2.2 Finding a Basis by Equivalence

If A and B are equivalent matrices, the following method can apply:

1. Use the vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ to form the columns of A.
2. $A \to B$ by Gauss elimination. The pivot columns of B are linearly independent, and the other columns are dependent.
3. The corresponding columns of A are the basis for S.

7.2.3 Dimension

Theorem 32. *If S is a subspace of $\mathbb{R}^n$, then every basis of S has the same number of vectors.*

In other words, let S be a subspace of $\mathbb{R}^n$. The dimension of S is the number of vectors in any basis of S.

Theorem 33. *Let $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a basis for a subspace of $S \neq \{0\} \subseteq \mathbb{R}^n$. Then either:*

- *If $\mathcal{U}$ is linearly independent, then either $\mathcal{U}$ is a basis for S or additional vectors can be added to $\mathcal{U}$ such that it is a basis for S.*
- *If $\mathcal{U}$ spans S, then either $\mathcal{U}$ is a basis for S, or vectors can be removed from $\mathcal{U}$ such that it is a basis for S.*

7.2.4 Nullity

The nullity of a matrix A is the dimension of the null space of A and is denoted nullity(A).

To find the nullity of a matrix, apply row reduction to $[A|0]$ and find the basis for the span.

Theorem 34. *Let $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a set of m vectors in S of dimension m. If $\mathcal{U}$ is either linearly independent or spans S, then it is a basis for S.*

Theorem 35. *Let $S_1 \subseteq S_2 \in \mathbb{R}^n$. Then:* $\dim(S_1) \leq \dim(S_2)$*, and* $\dim(S_1) = \dim(S_2) \iff S_1 = S_2$.

Theorem 36. *Let $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a set of vectors in a subspace S of dimension k. Then:*

1. *If $m < k$ then* $\text{span}(\mathcal{U}) \neq S$.
2. *If $m > k$ then $\mathcal{U}$ is not linearly independent.*

Therefore, we can amend The Big Theorem as follows:

Theorem 37. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. *S is linearly independent.*
3. $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\forall \mathbf{b} \in \mathbb{R}^n$.
4. *T is one-to-one, or injective.*
5. *T is onto, or surjective.*
6. *A is invertible.*
7. $\ker(T)$ *is* $\{\mathbf{0}\}$.
8. *S is a basis for* $\mathbb{R}^n$.

7.3 Row & Column Spaces

Suppose we have the following matrix, A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{7.2}$$

The *row space* of A (denoted $\text{row}(A)$) is defined as follows:

$$\text{row}(A) = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}, \cdots, \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \tag{7.3}$$

and the *column space* of A (denoted $\text{col}(A)$) is defined as follows:

$$\text{col}(A) = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \tag{7.4}$$

In summary:

Row space is the subspace of A spanned by the row vectors of A and is denoted $\text{row}(A)$.

Column space is the subspace of A spanned by the column vectors of A and is denoted $\text{col}(A)$.

Theorem 38. *Let $A \to B$ where B is in Echelon Form. Therefore:*

1. *The nonzero rows of B form a basis for* $\text{row}(A)$.
2. *The columns of A corresponding to the pivot columns of B correspond to form a basis for* $\text{col}(A)$.

Theorem 39. *For any matrix A:*

$$\dim(\text{row}(A)) = \dim(\text{col}(A)). \tag{7.5}$$

7.3.1 Rank

The rank of a matrix A is the dimension of the row- or column-space of A, and is denoted $\text{rank}(A)$.

To find the rank of a matrix A, apply Gaussian Elimination to A and count the number of non-zero rows.

Theorem 40. *Let $A_{n\times m}$. Then:*

$$\text{rank}(A) + \text{nullity}(A) = m. \tag{7.6}$$

Theorem 41. *Let A be an $n \times m$ matrix and $\mathbf{b} \in \mathbb{R}^n$.*

1. *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{col}(A)$.*
2. *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\mathbf{b} \in \text{col}(A)$ and $\text{col}(A)$ are linearly independent.*

Therefore, we may amend The Big Theorem as follows:

Theorem 42. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. *S is linearly independent.*
3. *$A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{R}^n$.*
4. *T is one-to-one, or injective.*
5. *T is onto, or surjective.*
6. *A is invertible.*
7. *$\ker(T)$ is $\{\mathbf{0}\}$.*
8. *S is a basis for $\mathbb{R}^n$.*
9. $\text{col}(A) = \mathbb{R}^n$.
10. $\text{row}(A) = \mathbb{R}^n$.
11. $\text{rank}(A) = n$.

Chapter 8

Determinants

8.1 The Determinant Function

Let the determinant of a matrix A be defined as follows:

$$\det(A_{m\times n}) = \sum_{k=1}^{n} = a_{1k}C_{1k}. \tag{8.1}$$

8.1.1 Minor

Let A be a matrix as follows:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{8.2}$$

Denote the ij-th minor M_{ij} as follows:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(j-1)} & a_{m(j+1)} & \cdots & a_{mn} \end{bmatrix} \tag{8.3}$$

8.1.2 Cofactor

Define the ij-th Cofactor of a matrix A as follows:

$$C_{ij} = (-1)^{i+j} \det(M_{ij}) \tag{8.4}$$

8.1.3 Special cases of det

There are the following special cases for consideration of det:

Theorem 43. $\forall n \geq 1 \to \det(I_n) = 1$.

Theorem 44. $\det(A)$*, where* A *is diagonal is the product of the diagonal entries.*

Theorem 45. $\det(A)$*, where* A *is an upper- or lower-triangular matrix is the product of the diagonal entries.*

Theorem 46. $\det(A^T) = \det(A)$.

Theorem 47. *If* A *has a row or column of zeros, or two equal rows or columns,* $\det(A) = 0$.

Theorem 48. $\det(AB) = \det(A)\det(B)$.

Theorem 49. *Let* A *be a square matrix.* A *is invertible if and only if* $\det(A) \neq 0$.

By the last theorem, we may extend The Big Theorem as follows:

Theorem 50. *Let* $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *be a set of* n *vectors in* $\mathbb{R}^n$*. Let* $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$*, and let* $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$*. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. S *is linearly independent.*
3. $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\forall \mathbf{b} \in \mathbb{R}^n$.
4. T *is one-to-one, or injective.*
5. T *is onto, or surjective.*

6. *A is invertible.*
7. $\ker(T)$ *is* $\{\mathbf{0}\}$.
8. *S is a basis for* $\mathbb{R}^n$.
9. $\text{col}(A) = \mathbb{R}^n$.
10. $\text{row}(A) = \mathbb{R}^n$.
11. $\text{rank}(A) = n$.
12. $\det(A) \neq 0$.

8.2 Properties of the Determinant

Theorem 51. *Let A be a square matrix.*

1. *Suppose that B is produced by interchanging two rows of A. Then* $\det(A) = -\det(B)$.
2. *Suppose that B is produced by multiplying a row of A by* $c \in \mathbb{R}$. *Then* $\det(a) = \frac{1}{c}\det(B)$.
3. *Suppose that B is produced by adding two rows of A to one another. Then* $\det(A) = \det(B)$.

8.3 Inverse Matrices via det

8.3.1 The Cofactor Matrix

Define the cofactor matrix C_A of A (denoted $C = \text{cof}(A)$) as follows:

$$C_A = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{m1} & \cdots & C_{mn} \end{bmatrix} \tag{8.5}$$

8.3.2 The Adjoint Matrix

Define the adjoint matrix of A (denoted $\mathrm{Adj}(A)$) as follows:

$$\mathrm{Adj}(A) = (\mathrm{cof}(A))^T \tag{8.6}$$

Note that:

$$\mathrm{Adj}(A)A = \det(A)I_n. \tag{8.7}$$

or, stated more simply:

$$\frac{1}{\det(A)}\,\mathrm{Adj}(A) = A^{-1}. \tag{8.8}$$

Chapter 9

Eigenvalues & Eigenvectors

9.1 Eigenvalues & Eigenvectors

Given a linear transformation $T(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n = A\mathbf{x}$, we denote an eigenvalue λ such that $\exists \mathbf{x}, \lambda.\ T(\mathbf{x}) = \lambda\mathbf{x}$.

1. We denote $\mathbf{x}$ in the above to be the "eigenvector".
2. We denote λ in the above to be the "eigenvalue".

Theorem 52. *Suppose that $A_{n\times n}$ and $\mathbf{x}$ is an eigenvector of A assosciated to the eigenvalue λ. Then for $c \in \mathbb{R}, c \neq 0$, $c\mathbf{x}$ is also an eigenvector assosciated to A.*

9.1.1 Finding Eigenvalues

Suppose that $A_{n\times n}$. λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

See: characteristic polynomials; pre-calculus.

Denote the multiplicity of λ as the exponent of the factor $(x - \lambda)$ in the characteristic polynomial.

9.1.2 Finding Eigenvectors

Suppose that we are given $A_{n\times n}$ and an eigenvalue λ assosciated to A. We can then solve for an eigenvector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

1. Express the left-hand of the above as a matrix equation.

2. Express the right-hand of the above as a vector-scalar quantity.

3. Equate the two, and simplify towards a homogenous system.

4. Solve using existing methods. A basis for the solution set (E_λ) will be a vector space for which all values are eigenvectors.

9.1.3 Bounding Eigenspaces

Theorem 53. *Given $A_{n\times n}$, and an eigenvalue λ assosicated to A, the dimension of the eigenspace E_λ is bounded as follows:*

$$1 \leq \dim(E_\lambda) \leq \text{multiplicity}(\lambda) \tag{9.1}$$

Theorem 54. *Consider a matrix $A_{n\times n}$ and a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n = A(\mathbf{x})$. T is injective if and only if 0 is not an eigenvalue.*

9.2 Diagonalization

Consider a matrix $A_{n\times n}$. A is said to be "diagonalizable" if and only if:

$$\exists P, D.\ A = PDP^{-1}. \tag{9.2}$$

Where the above matrix P is invertible, and the above matrix D is diagonal (i.e., only contains entries at $D_{i=j}$).

9.2.1 Diagonalizing

Suppose that the matrix $A_{n\times n}$ has eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, apply the following:

1. Let $P = [\mathbf{x}_1 \cdots \mathbf{x}_n]$.

2. Let $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

It follows from the above that $AP = PD$. Since A^{-1} exists, take the outer product of the two expressions to obtain $APP^{-1} = PDP^{-1}$, and simplify to:

$$A = PDP^{-1}. \tag{9.3}$$

Remark: in the above, denote the diag operation as follows:

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{9.4}$$

9.3 Change of Basis Matrix

Notice that for any basis of $\mathbb{R}^n$, any vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed as:

$$\sum_{i=0}^{n} x_i \mathbf{e}_i. \tag{9.5}$$

but for any alternative basis of $\mathbb{R}^n$, say B, we note that each $\mathbf{x} \in \mathbb{R}^n$ can be expressed in terms of the components of B, as follows:

$$\sum_{i=0}^{n} x'_i \mathbf{b}_i. \tag{9.6}$$

We denote the above summation as $\mathbf{x}_B$ (the vector $\mathbf{x}$ written in B) as $\mathbf{x}_B = (x'_1, \ldots, x'_n)^T$.

9.3.1 Remarks on $M_{B_1 \to B_2}$

Notice first the composition of the above matrix $M_{B_1 \to B_2}$.

- From B_1 to B_3 through B_2:

$$M_{B_2 \to B_3} \cdot M_{B_1 \to B_2} = M_{B_1 \to B_3}. \tag{9.7}$$

- From B_n to B_n:

$$M_{B_n \to B_n} = I_n. \tag{9.8}$$

- On the invertibility of M:

$$M_{B_n \to B_m} = B^{-1}_{B_m \to B_n}. \tag{9.9}$$

9.3.2 Finding $M_{B_2 \to B_1}$

Given B_1 and B_2, find a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n : T(\mathbf{x}_{B_2}) = \mathbf{x}_{B_1}$:

1. Write $M_{B_1 \to S.B.}$, and invert to find the transformation from the standard basis into B_1, $M^{-1}_{B_1 \to S.B.}$.
2. Write $M_{B_2 \to S.B.}$ by writing the basis vectors $M_{B_2 \to S.B.} = [\mathbf{b}_1 \cdots \mathbf{b}_n]$.
3. Compose the two matricies as:

$$M_{B_2 \to B_1} \cdot M^{-1}_{B_1 \to S.B.} = M_{B_2 \to S.B.}. \tag{9.10}$$