MATH308 A, Eufemia, Exam II

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[1]Material is compiled from the lecture notes, the readings in *Linear Algebra with Applications* (2nd edition), Holt, and my own notes.

Contents

Chapter 4

Subspaces

4.1 Introduction to Subspaces

A subset S of $\mathbb{R}^n$ is a subspace if it satisfies the following three conditions:

1. $\mathbf{0} \in S$.
2. $\mathbf{u}, \mathbf{v} \in S \to \mathbf{u} + \mathbf{v} \in S$. i.e., S is closed over scalar addition.
3. $r \in \mathbb{R}, \mathbf{u} \in S \to r\mathbf{u} \in S$. i.e., S is closed over scalar multiplication.

4.1.1 Classification of Subspaces

To classify whether a set S is a subspace $S \subseteq \mathbb{R}^n$, first begin by checking whether $\mathbf{0} \in S$. Then check if two "obvious" vectors included in S are also found in the sum of their vectors. If the set S is closed over addition, check again for scalar multiples. If no violations are found, then S is a subspace of $\mathbb{R}^n$.

Theorem 1. *Let* $S = \text{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$. *Then* S *is a subspace of* $\mathbb{R}^n$.

4.1.2 null-spaces

Let A be an $n \times m$ matrix. The set of solutions to $A\mathbf{x} = \mathbf{0}$ is a null space and is denoted by $\text{null}\, A$.

Theorem 2. *Let* A *be an* $n \times m$ *matrix.* $\text{null}(A) = S$, S *is a subspace of* $\mathbb{R}^m$.

4.1.3 ker, & range of T

image(T) $\{\mathbf{y} \mid \exists \mathbf{x} : T(\mathbf{x}) = \mathbf{y}\}$.

ker(T) $\{\mathbf{x} \mid T(\mathbf{x}) = \mathbf{0}\}$.

In other terms, the image (image) operation on a linear transformation T is everything in the output domain of T for which there exists an input such that T will transform it to the output.

Likewise, the kernel of T (denoted ker) is the set of all input vectors to T that are transformed under T to $\mathbf{0}$.

Theorem 3. *Let $T : \mathbb{R}^m \to \mathbb{R}^n$.*

1. ker(T) *is a subspace of the domain in $\mathbb{R}^m$.*

2. image(T) *is a subspace of the subset of the codomain in $\mathbb{R}^n$.*

Theorem 4. *Let $T : \mathbb{R}^m \to \mathbb{R}^n$. T is one-to-one if and only if* $\ker(T) = \{\mathbf{0}\}$.

Therefore, we may amend The Big Theorem to include the above theorem:

Theorem 5. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. span$\{S\} = \mathbb{R}^n$.

2. *S is linearly independent.*

3. *$A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{R}^n$.*

4. *T is one-to-one, or injective.*

5. *T is onto, or surjective.*

6. *A is invertible.*

7. ker(T) *is* $\{\mathbf{0}\}$.

4.2 Basis and Dimension

Denote $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ as the basis for a subspace S if the following are met:

1. $\operatorname{span} \mathcal{B} = S$.
2. $\mathcal{B}$ is linearly independent.

Theorem 6. *Let $\mathcal{B} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a basis for a subspace S. Then:*

$$\forall \mathbf{s} \in S \mid \exists s_1, \ldots, s_m : \mathbf{s} = \sum_{i=0} s_i \mathbf{u_i}. \tag{4.1}$$

4.2.1 Finding a Basis

To find a basis for a set of vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$, apply Gauss elimination on a matrix A where each *row* is $\mathbf{u_i}$, and the non-zero rows are the basis $\mathcal{B}$ of S.

4.2.2 Finding a Basis by Equivalence

If A and B are equivalent matrices, the following method can apply:

1. Use the vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ to form the columns of A.
2. $A \to B$ by Gauss elimination. The pivot columns of B are linearly independent, and the other columns are dependent.
3. The corresponding columns of A are the basis for S.

4.2.3 Dimension

Theorem 7. *If S is a subspace of $\mathbb{R}^n$, then every basis of S has the same number of vectors.*

In other words, let S be a subspace of $\mathbb{R}^n$. The dimension of S is the number of vectors in any basis of S.

Theorem 8. *Let $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a basis for a subspace of $S \neq \{0\} \subseteq \mathbb{R}^n$. Then either:*

- *If $\mathcal{U}$ is linearly independent, then either $\mathcal{U}$ is a basis for S or additional vectors can be added to $\mathcal{U}$ such that it is a basis for S.*

- *If $\mathcal{U}$ spans S, then either $\mathcal{U}$ is a basis for S, or vectors can be removed from $\mathcal{U}$ such that it is a basis for S.*

4.2.4 Nullity

The nullity of a matrix A is the dimension of the null space of A and is denoted nullity(A).

To find the nullity of a matrix, apply row reduction to $[A|0]$ and find the basis for the span.

Theorem 9. *Let $\mathcal{U} = \{\mathbf{u_1}, \dots, \mathbf{u_m}\}$ be a set of m vectors in S of dimension m. If $\mathcal{U}$ is either linearly independent or spans S, then it is a basis for S.*

Theorem 10. *Let $S_1 \subseteq S_2 \in \mathbb{R}^n$. Then:* $\dim(S_1) \leq \dim(S_2)$, *and* $\dim(S_1) = \dim(S_2) \iff S_1 = S_2$.

Theorem 11. *Let $\mathcal{U} = \{\mathbf{u_1}, \dots, \mathbf{u_m}\}$ be a set of vectors in a subspace S of dimension k. Then:*

1. *If $m < k$ then* $\text{span}(\mathcal{U}) \neq S$.
2. *If $m > k$ then $\mathcal{U}$ is not linearly independent.*

Therefore, we can amend The Big Theorem as follows:

Theorem 12. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \dots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. *S is linearly independent.*
3. *$A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{R}^n$.*
4. *T is one-to-one, or injective.*
5. *T is onto, or surjective.*
6. *A is invertible.*
7. $\ker(T)$ *is* $\{\mathbf{0}\}$.
8. *S is a basis for $\mathbb{R}^n$.*

4.3 Row & Column Spaces

Suppose we have the following matrix, A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{4.2}$$

The *row space* of A (denoted $\text{row}(A)$) is defined as follows:

$$\text{row}(A) = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}, \cdots, \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \tag{4.3}$$

and the *column space* of A (denoted $\text{col}(A)$) is defined as follows:

$$\text{col}(A) = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \tag{4.4}$$

In summary:

Row space is the subspace of A spanned by the row vectors of A and is denoted $\text{row}(A)$.

Column space is the subspace of A spanned by the column vectors of A and is denoted $\text{col}(A)$.

Theorem 13. *Let $A \to B$ where B is in Echelon Form. Therefore:*

1. *The nonzero rows of B form a basis for* $\text{row}(A)$.

2. *The columns of A corresponding to the pivot columns of B correspond to form a basis for* $\text{col}(A)$.

Theorem 14. *For any matrix A:*

$$\dim(\text{row}(A)) = \dim(\text{col}(A)). \tag{4.5}$$

4.3.1 Rank

The rank of a matrix A is the dimension of the row- or column-space of A, and is denoted $\text{rank}(A)$.

To find the rank of a matrix A, apply Gaussian Elimination to A and count the number of non-zero rows.

Theorem 15. *Let $A_{n\times m}$. Then:*

$$\text{rank}(A) + \text{nullity}(A) = m. \tag{4.6}$$

Theorem 16. *Let A be an $n \times m$ matrix and $\mathbf{b} \in \mathbb{R}^n$.*

1. *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{col}(A)$.*
2. *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\mathbf{b} \in \text{col}(A)$ and $\text{col}(A)$ are linearly independent.*

Therefore, we may amend The Big Theorem as follows:

Theorem 17. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. $\text{span}\{S\} = \mathbb{R}^n$.
2. *S is linearly independent.*
3. *$A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{R}^n$.*
4. *T is one-to-one, or injective.*
5. *T is onto, or surjective.*
6. *A is invertible.*
7. *$\ker(T)$ is $\{\mathbf{0}\}$.*
8. *S is a basis for $\mathbb{R}^n$.*
9. $\text{col}(A) = \mathbb{R}^n$.
10. $\text{row}(A) = \mathbb{R}^n$.
11. $\text{rank}(A) = n$.

Chapter 5

Determinants

5.1 The Determinant Function

Let the determinant of a matrix A be defined as follows:

$$\det(A_{m\times n}) = \sum_{k=1}^{n} = a_{1k}C_{1k}. \tag{5.1}$$

5.1.1 Minor

Let A be a matrix as follows:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{5.2}$$

Denote the ij-th minor M_{ij} as follows:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(j-1)} & a_{m(j+1)} & \cdots & a_{mn} \end{bmatrix} \tag{5.3}$$

5.1.2 Cofactor

Define the ij-th Cofactor of a matrix A as follows:

$$C_{ij} = (-1)^{i+j}\det(M_{ij}) \tag{5.4}$$

5.1.3 Special cases of det

There are the following special cases for consideration of det:

Theorem 18. $\forall n \geq 1 \rightarrow \det(I_n) = 1$.

Theorem 19. $\det(A)$, *where A is diagonal is the product of the diagonal entries.*

Theorem 20. $\det(A)$, *where A is an upper- or lower-triangular matrix is the product of the diagonal entries.*

Theorem 21. $\det(A^T) = \det(A)$.

Theorem 22. *If A has a row or column of zeros, or two equal rows or columns,* $\det(A) = 0$.

Theorem 23. $\det(AB) = \det(A)\det(B)$.

Theorem 24. *Let A be a square matrix. A is invertible if and only if* $\det(A) \neq 0$.

By the last theorem, we may extend The Big Theorem as follows:

Theorem 25. *Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\mathbf{a}_1, \ldots \mathbf{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n = T(\mathbf{x}) = A\mathbf{x}$. The following are therefore equivalent:*

1. $\operatorname{span}\{S\} = \mathbb{R}^n$.

2. *S is linearly independent.*

3. *$A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{R}^n$.*

4. *T is one-to-one, or injective.*

5. *T is onto, or surjective.*

6. *A is invertible.*
7. $\ker(T)$ *is* $\{\mathbf{0}\}$.
8. *S is a basis for* $\mathbb{R}^n$.
9. $\operatorname{col}(A) = \mathbb{R}^n$.
10. $\operatorname{row}(A) = \mathbb{R}^n$.
11. $\operatorname{rank}(A) = n$.
12. $\det(A) \neq 0$.

5.2 Properties of the Determinant

Theorem 26. *Let A be a square matrix.*

1. *Suppose that B is produced by interchanging two rows of A. Then* $\det(A) = -\det(B)$.
2. *Suppose that B is produced by multiplying a row of A by* $c \in \mathbb{R}$. *Then* $\det(a) = \frac{1}{c}\det(B)$.
3. *Suppose that B is produced by adding two rows of A to one another. Then* $\det(A) = \det(B)$.

5.3 Inverse Matrices via det

5.3.1 The Cofactor Matrix

Define the cofactor matrix C_A of A (denoted $C = \operatorname{cof}(A)$) as follows:

$$C_A = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{m1} & \cdots & C_{mn} \end{bmatrix} \tag{5.5}$$

5.3.2 The Adjoint Matrix

Define the adjoint matrix of A (denoted $\mathrm{Adj}(A)$) as follows:

$$\mathrm{Adj}(A) = (\mathrm{cof}(A))^T \tag{5.6}$$

Note that:

$$\mathrm{Adj}(A)A = \det(A)I_n. \tag{5.7}$$

or, stated more simply:

$$\frac{1}{\det(A)}\,\mathrm{Adj}(A) = A^{-1}. \tag{5.8}$$

Chapter 6

Miscellany

1. Kernels, Range, and Null-, Row-, and Column-spaces of a matrix.
2. Determinant definition, and cofactor expansion.
3. Computing the basis of the intersection and sum of two linear subspaces.
4. Algorithms to obtain projections and reflections.