MATH 309 B, Yuan

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[1]Material is compiled from the lecture notes, the readings in *Linear Anyalysis* (10th edition), Boyce, and my own notes. The majority of material is transcribed from Dr. Yuan's notes, and can be found in their original form on our course webpage.

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Chapter 2

Systems of first order linear differential equations

2.1 Real Eigenvalues, the matrix exponential

Example 2.1.1. Suppose that we are considering the following first-order system of linear differential equations:

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

Recall that this looks analytically quite similar to the 1-dimensional case:

$$x' = cx$$

which is solved by:

$$\int \frac{1}{x} ct = \int c\, ct$$

or

$$\ln x = ct \iff x = Ae^{ct}$$

So, how can we exponentiate by a matrix power? Let us define the matrix $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and consider e^{Jt}. We will do so by recalling that:

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}$$

So, we see that:

$$e^{Jt} = \sum_{n=0}^{\infty} \frac{J^n t^n}{n!}$$

Considering powers of J, we see that:

$$J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J^3 = -J, \quad J^4 = I$$

or that:

$$\begin{aligned} e^{Jt} &= I + \frac{Jt}{1!} + \frac{(Jt)^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{5!}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n \frac{n^{2n}}{(2n)!} & \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^{2n-1}}{(2n-1)!} \\ \sum_{n=1}^{\infty}(-1)^n \frac{n^{2n-1}}{(2n-1)!} & \sum_{n=0}^{\infty}(-1)^n \frac{n^{2n}}{(2n)!} \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

Example 2.1.2. Now, let's consider a case that doesn't have a pattern that "pops out" as easily as the example e^{Jt}. Let's consider:

$$x' = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} x$$

Notice that we have $\lambda_1 = 2$, $\lambda_2 = -1$, with $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Call the matrix formed by the eigenvectors $P = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$, and D the diagonal matrix of eigenvalues: $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Observe that:

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

or that:

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^{-1} = PDP^{-1}$$

Now recall the following property from linear algebra:

$$(PDP^{-1})^k = PD^kP^{-1}$$

apply that to the matrix exponential case:

$$(Mt)^k = P\begin{bmatrix} (2t)^k & 0 \\ 0 & (-t)^k \end{bmatrix} P^{-1}$$

so we have:

$$\begin{aligned} e^{Mt} &= I + Mt + \frac{(Mt)^2}{2!} + \cdots + \frac{(Mt)^n}{n!} \\ &= PIP^{-1} + P(Dt)P^{-1} + \frac{P(Dt)^2P^{-1}}{2!} + \cdots + \frac{P(Dt)^nP^{-1}}{n!} \\ &= P\begin{bmatrix} 1 + 2t + \frac{(2t)^2}{2!} + \cdots + \frac{(2t)^n}{n!} & 0 \\ 0 & 1 + (-t) + \frac{(-t)^2}{2!} + \cdots + \frac{(-t)^n}{n!} \end{bmatrix} P^{-1} \\ &= P\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} P^{-1} \end{aligned}$$

Example 2.1.3. Now let's consider the system:

$$x' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$$

The "matrix exponential" route gives us that:

$$x = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^{-1} x(0)$$

and that if we let:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^{-1} x(0)$$

then we have:

$$\begin{aligned} x &= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 \end{aligned}$$

We can take another, more geometric, approach, too. Observe that:

$$\begin{aligned} M &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - I \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} - I \end{aligned}$$

Consider $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Note that:

$$\begin{aligned} Mv_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - I \\ &= 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - I \\ &= 2v_1 \end{aligned}$$

By inspection, we obtain that two perpendicular vectors v_2, v_3 are:

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

which both have $\lambda_{2,3} = -1$

2.2 Complex Eigenvalues

Example 2.2.1. Consider the system:

$$x' = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} x$$

Let us solve it in two ways. First, we'll consider the eigenvalue route. Observe that we have $x' = Mx$, and M has $\lambda_{1,2} = \frac{-1}{2} \pm i$, with $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Then, we have:

$$x = \underbrace{\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{\left(-\frac{1}{2}+i\right)t} & 0 \\ 0 & e^{\left(-\frac{1}{2}-i\right)t} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}}_{\text{Real matrix}} x(0)$$

Let us consider another approach, which is to note that:

$$x' = \left(\begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) x$$

and that immediately we have:

$$\begin{aligned} x &= e^{\left(-\frac{1}{2}I+J\right)t} x(0) = e^{-\frac{1}{2}It} e^{Jt} x(0) \\ &= e^{-\frac{1}{2}t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x(0) \end{aligned}$$

Example 2.2.2. Let us consider another approach, to show that the solutions of systems of ODEs with complex eigenvalues are the linear combination of the Re and Im part along each eigenvector.

$$x' = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} x$$

Observe that we have $\lambda_{1,2} = \pm 3i\sqrt{3}$. Note that we can pick either $\lambda_1 > 0$, or $\lambda_2 < 0$ to obtain v_1 or v_2, respectively, and they are complex conjugates of one another. For example, let's consider λ_1, and note that we obtain the system:

$$\begin{aligned} 9x_2 &= \left(-3 + 3\sqrt{3}i\right) x_1 \\ x_2 &= -\frac{1}{3}\left(1 - \sqrt{3}i\right) x_1 \end{aligned}$$

which is solved by:

$$x^{(1)} = \begin{pmatrix} 3 \\ \sqrt{3}i - 1 \end{pmatrix}, \quad x^{(2)} = \overline{x^{(1)}} = \begin{pmatrix} 3 \\ -\sqrt{3}i - 1 \end{pmatrix}$$

Consider the partial solutions:

$$\tilde{x}(t) = e^{3\sqrt{3}it} \begin{pmatrix} 3 \\ \sqrt{3}i - 1 \end{pmatrix} = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) + 3i\sin\left(3\sqrt{3}t\right) \\ \sqrt{3}i\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) - \cos\left(3\sqrt{3}t\right) - i\sin\left(3\sqrt{3}t\right) \end{pmatrix}$$

and note that the full solution is comprised of $x(t) = u(t) + iv(t)$, or that:

$$x(t) = c_1 \operatorname{Re}(\tilde{x}(t)) + c_2 \operatorname{Im}(\tilde{x}(t))$$

2.3 Repeated Eigenvalues

Example 2.3.1. Suppose we have the system:

$$x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x, \quad \text{subject to } x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

Consider powers of M. Note that $M^0 = I$, $M^1 = M$, and that M^k for $k \geq 2$ vanishes. So, we have:

$$e^{Mt} = \sum_{k=0}^{\infty} \frac{(Mt)^k}{k!} = I + Mt$$

and thus:

$$x(t) = e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} x(0) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x(0)$$

In the next two example, we consider:

$$x' = Ax = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$$

subject to $x(0)$.

Example 2.3.2. One possible approach is to diagonalize according to Jordan form. First, we find eigenvalues λ, according to:

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

Hint: when $\lambda_1 = \lambda_2 = 0$, then A^k for $k > 0$ is $0_{2\times 2}$.[1]

But how do we apply this hint to our case? Note that:

$$\underbrace{A - 2I}_{\lambda_{1,2}=0} + 2I = A$$

Now, we can use the matrix exponential to find the fundamental matrix, with:

$$\begin{aligned} e^{At} &= e^{(A-2I)t + 2It} = e^{2t} e^{(A-2I)t} \\ &= \sum_{k=0}^{\infty} \frac{(A-2I)^k t^k}{k!} \\ &= I + \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} t \\ &= \begin{pmatrix} 1-t & -t \\ t & 1-1 \end{pmatrix} \end{aligned}$$

[1] In general, for arbitrary dimension $\mathbb{R}^{k\times k}$, if $\lambda_1 = 0$ with algebraic multiplicity k, then $A^l = 0_{k\times k}$ for $l \geq k$.

and then:

$$x(t) = e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1-1 \end{pmatrix} x(0)$$

Example 2.3.3. Let us consider another approach, which is to take the Jordan-form diagonalization of A as follows. First, observe that we can find easily that $\lambda_{1,2} = 2$ and that $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then, we have to find w_2 such that $(A - 2I)w_2 = v_1$.

By inspection, we see that w_2 is given as:

$$w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Observe now that we have:

$$\begin{aligned} Av_1 &= 2v_1 \\ Aw_2 &= 2w_2 + v_1 \end{aligned}$$

or that:

$$A = \begin{pmatrix} v_1 & w_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 & w_2 \end{pmatrix}^{-1}$$

and thus:

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= Pe^{(D-2I)t+2It}P^{-1} \\ &= Pe^{2t} \left(\sum_{k=1}^{\infty} \frac{((D-2I)t)^k}{k!} \right) P^{-1} \\ &= Pe^{2t} \left[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right] P^{-1} \end{aligned}$$

and we have a general solution:

$$\begin{aligned} x(t) &= e^{At}x(0) \\ &= \begin{pmatrix} e^{2t}v_1 & e^{2t}w_2 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 & w_2 \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t}v_1 & e^{2t}w_2 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{c_1} \\ \tilde{c_2} \end{pmatrix} \end{aligned}$$

2.4 Non-homogeneous linear systems

Recall our treatment of non-homogeneous systems in the scalar case. If given a system as:

$$\begin{cases} u'(t) = mu + f(t) \\ u(0) = a \end{cases}$$

we solved it as:

$$u(t) = ae^{mt} + e^{mt} \int_0^t e^{-ms} f(s)\, ds$$

Now, we transform to a n-dimensional *system* of linear ODEs, of the form:

$$X' = Mx + F(t), \quad \text{subject to } x(0)$$

which is similarly solved as:

$$x(t) = e^{Mt}x(0) + e^{Mt}\int_0^t e^{-Ms}F(s)\,ds$$

Example 2.4.1. Let's consider the following example:

$$x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

One can easily verify that we have $\lambda_1 = -1, \lambda_2 = -3$, with $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Now we find the fundamental matrices Φ:

$$e^{Mt} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}$$
$$e^{-Ms} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{s} & 0 \\ 0 & e^{3s} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}$$

and solve for $x(t)$:

$$x(t) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-3t} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1} x(0) + P \begin{pmatrix} e^{-t} & \\ & e^{-3t} \end{pmatrix} \cancel{P^{-1}} \int_0^t \cancel{P} \begin{pmatrix} e^{-s} & \\ & e^{-3s} \end{pmatrix} P^{-1} \begin{pmatrix} 2e^{-s} \\ 3s \end{pmatrix} ds$$

we compute the integral as:

$$\begin{aligned}
&\int_0^t \begin{pmatrix} e^{-s} & \\ & e^{-3s} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-s} \\ 3s/2 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} e^{s} & e^{s} \\ e^{3s} & -e^{3s} \end{pmatrix} \begin{pmatrix} e^{-s} \\ 3s/2 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} 1 + 3se^{s}/2 \\ e^{2s} - 3se^{3s}/2 \end{pmatrix} ds \\
&= \begin{pmatrix} s + \frac{3}{2}se^{s} - \frac{3}{2}e^{s} \\ \frac{1}{2}e^{2s} - \frac{1}{2}se^{3s} + \frac{1}{6}e^{3s} \end{pmatrix} \Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \\ \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \end{pmatrix} - \begin{pmatrix} -3/2 \\ 2/5 \end{pmatrix}
\end{aligned}$$

And thus have:

$$x(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-3t} \end{pmatrix} \begin{pmatrix} t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + \frac{3}{2} \\ \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} - \frac{2}{5} \end{pmatrix}$$

Chapter 3

Partial Differential Equations & Fourier Series

3.1 Fourier Series

First, recall the following basic formulas from trigonometry:

$$\begin{aligned} e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha + beta) \\ e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + cos\alpha\sin\beta) \end{aligned}$$

$$\begin{aligned} \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} &= \sin\alpha\sin\beta \\ \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} &= \cos\alpha\cos\beta \\ \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} &= \sin\alpha\cos\beta \end{aligned}$$

Example 3.1.1. Consider:

$$\begin{cases} u_{tt} = u_{xx} \\ u_t(x,0) = 0 \\ u(x,0) = \sin x \end{cases}$$

We solve by "guessing" a solution of the form $u = \sin(x)T(t)$. Observe that:

$$u_{tt} = T''(t)\sin(x), \quad u_{xx} = -\sin(x)T(t)$$

So we have that $T(t) = -T''(t)$, and thus that $T(t) = \sin t$, or $\cos t$.[1]

[1] In general, we have that solutions to the wave equation will be of the form:

$$c_1\sin(x)\sin(t) + c_2\sin(x)\cos(t).$$

Now, we check our boundary conditions as follows:

$$\begin{aligned} u(x,0) &= c_1 \sin(x)\sin(0) + c_2 \sin(x)\cos(0) \\ &= c_2 \sin(x) \qquad\qquad \Rightarrow c_2 = 1 \end{aligned}$$

$$\begin{aligned} u_t(x,0) &= c_1 \sin(x)\cos(t) - c_2 \sin(x)\sin(t)\big|_{t=0} \\ &= c_1 \sin(x)\cos(0) - (1)\sin(x)\sin(0) \\ &= c_1 \sin(x) \qquad\qquad \Rightarrow c_1 = 0 \end{aligned}$$

This, we have:

$$u(x,t) = \sin(x)\cos(t)$$

Example 3.1.2. Let us now consider an example of the form:

$$\begin{cases} u_{tt} = u_{xx} \\ u_t(x,0) = 0 \\ u(x,0) = \sin kx \end{cases}$$

Again, we have:

$$u(x,t) = c_1 \sin(2x)\sin(2t) + c_2 \sin(2x)\cos(2t)$$

and by a similar process as above (to isolate c_1, c_2), we have in general that:

$$u(x,t) = \sin(kx)\cos(kt)$$

Likewise, for linear combinations of sin-functions, we have that if, say:

$$u(x,0) = \sin x + 5\sin(2x) - e\sin(7x)$$

that:

$$u(x,t) = \sin(x)\cos(t) + 5\sin(2x)\cos(2t) - e\sin(7x)\cos(7t)$$

Example 3.1.3. Now, we consider the example of a triangle wave on the period $[0,\pi]$. We denote this function as:

$$\wedge(x) = \begin{cases} x & x \in [0,\pi/2] \\ \pi - x & x \in [\pi/2,\pi] \end{cases}$$

We wish to decompose $\wedge$, such that:

$$\wedge(x) = \sum_{k=1}^{\infty} b_k \sin(kx) = b_1 \sin(kx) + \cdots + b_k \sin(kx)$$

Such that we can match $\cos(kt)$ terms to each of the $\sin(kt)$ terms to solve a wave equation where the initial displacement $u(x,0) = \wedge(x)$.

$$\begin{aligned} \int_0^{\pi} \wedge(x)\sin(kx)dx &= \int_0^{\pi} b_1 \sin(x)\sin(kx) + \cdots + b_k \sin^2(kx)dx \\ \int_0^{\pi/2} x\sin(kx)dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(kx)dx &= b_1 \frac{\frac{\sin(x-kx)}{1-k} - \frac{\sin(x+kx)}{1+k}}{2} + \cdots + b_k \frac{x - \frac{\sin(2kx)}{2k}}{2}\Bigg|_{x=0}^{\pi} \\ &= \frac{\pi}{2} b_k \end{aligned}$$

We solve the left-hand side as:

$$\begin{aligned}
\cdots &= \left(-x\frac{\cos(kx)}{k} + \frac{\sin(kx)}{k^2}\right)\Big|_{x=0}^{\pi/2} + \left(-(\pi - x)\frac{\cos(kx)}{k} - \frac{\sin(kx)}{k^2}\right)\Big|_{x=\pi/2}^{\pi} \\
&= -\frac{\pi}{2}\frac{\cos(k\pi/2)}{2} + \frac{\sin(k\pi/2)}{k^2} + \frac{\pi}{2}\frac{\cos(k\pi/2)}{2} + \frac{\sin(k\pi/2)}{k^2} \\
&= \frac{2\sin(k\pi/2)}{k^2} \\
&= \begin{cases} \frac{2(-1)^n}{k^2} & k = 2n+1, \text{ odd} \\ 0 & k \text{ even} \end{cases}
\end{aligned}$$

Together, we have that:

$$b_k = \begin{cases} \frac{4(-1)^n}{k^2\pi} & k \,\text{odd} \\ 0 & \text{otherwise} \end{cases}$$

and that:

$$\wedge(x) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)^2\pi} \sin((2n+1)x)\cos((2n+1)t)$$

Example 3.1.4. Solve the wave equation which is given as:

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t), \; x \in [0,\pi], t > 0 \\ v_t(x,0) = g(x), \; x \in [0,\pi] \\ v(x,0) = 0 \end{cases}$$

First, note that when $g(x) = \sin kx$ that our job is easy; the general solution is:

$$c_1 \sin(kx)\cos(kt) + c_2 \sin(kx)\sin(kt)$$

We find that $v(x,0) = 0$ implies that $c_1 = 0$, and that $v_t(x,0) = \sin(kx)$ implies $c_2 = \frac{1}{k}$. Therefore, for integer k, we have:

$$v(x,t) = \sin(kx)\frac{\sin(kt)}{k}$$

When $g(x)$ is more "complex", we take the Fourier expansion in terms of $\sin(kx)$, such that $g(x) = \sum_{i=1}^{\infty} b_k \sin(kx)$. Recall the Euler-Fourier formula:

$$b_k = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(kx)\,dx$$

and then we have:

$$v(x,t) = \sum_{k=1}^{\infty} b_k \sin(kx)\frac{\sin(kt)}{k}$$

Example 3.1.5 ($\star$)**.** Solve the wave equation which is given as:

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t),\ x \in [0,\pi], t > 0 \\ v_t(x,0) = g(x),\ x \in [0,\pi] \\ v(x,0) = f(x),\ x \in [0,\pi] \end{cases}$$

We take the sine-Fourier expansion of *both* f, g as:

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx), \text{ where } c_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$$
$$g(x) = \sum_{k=1}^{\infty} b_k \sin(kx), \text{ where } b_k = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(kx)\,dx$$

Then, the solution $v(x,t)$ is:

$$v(x,t) = \sum_{k=1}^{\infty} c_k \sin(kx)\cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kx)\frac{\sin(kt)}{k}$$

Example 3.1.6. Now we turn our attention to the *heat equation* over a closed loop, which is governed by:

$$\begin{cases} h_t(\theta,t) = h_{\theta\theta}(\theta,t), \quad \theta \in [0,2\pi],\ t > 0 \\ h(\theta,0) = f(\theta), \quad \theta \in [0,2\pi] \end{cases}$$

First, begin by noting that:

- In a closed loop, there is no boundary condition.
- The initial velocity $h_t(\theta,0) = h_{\theta\theta}(\theta,0) = f_{\theta\theta}(\theta)$ is determined by the initial temperature $f(\theta)$.

We can consider a few representative cases:

$f(\theta) = \cos\theta$ If $h(\theta,t) = \cos(\theta)T(t)$, then

$$h_t(\theta,t) = \cos(\theta)T'(t)$$

and

$$h_{\theta\theta} = -\cos(\theta)T(t)$$

Then, we have $T'(t) = -T(t)$, and surmise that $T(t) = e^{-t}$. Thus, we have that:

$$h(\theta,t) = \cos(\theta)e^{-t}$$

$f(\theta) = \frac{1}{3}\sin(5\theta)$ By a similar reasoning as above, if we let:

$$h(\theta,t) = \frac{1}{3}\sin(5\theta)T(t)$$

then

$$h_t = \frac{1}{3}\sin(5\theta)T'(t)$$

and

$$h_{\theta\theta} = -\frac{1}{3}5^2\sin(5\theta)T(t)$$

So we have that $-5^2T(t) = T'(t)$, and thus that $T(t) = e^{-5^2t}$.

Example 3.1.7 ($\star$)**.** In general, let us consider the heat equation on a closed loop with 2π-periodicity, such that:

$$\begin{cases} h_t(\theta,t) = h_{\theta\theta}(\theta,t), & \theta \in [0,2\pi],\ t > 0 \\ h(\theta,0) = f(\theta), & \theta \in [0,2\pi] \end{cases}$$

Then we take the Fourier-series expansion of f, such that:

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(k\theta) + b_k\sin(k\theta)$$

where:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,d\theta \quad k \in \{0,1,2,\dots\}$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta)\,d\theta \quad k \in \{1,2,\dots\}$$

Thus by superposition, and the appropriate scaling factor, we have:

$$h(\theta,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(k\theta) + b_k\sin(k\theta)\right)e^{-k^2t}$$

3.2 Heat Conduction, Diffusion

Example 3.2.1. Consider a wire of length ℓ (i.e., a circle embedding with $r = \ell/2\pi$), such that:

$$\begin{cases} h_t(s,t) = h_{ss}(s,t), & s \in [0,\ell],\ t > 0 \\ h(s,0) = g(s) \end{cases}$$

Where $g(s) = s^2$ on $[-\ell/2, \ell/2]$.

Observe that the Fourier expansion of $g(s)$ (with $\ell = 4$) is:

$$g(s) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(k\frac{\pi}{2}s\right)$$

By matching, we obtain:

$$h(s,t) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(k\frac{\pi}{2}s\right)e^{\left(-k\frac{\pi}{2}\right)^2t}$$

Observe that:

$$\lim_{t\to\infty} h(s,t) = \frac{4}{3}$$

Since all of the $e^{(-k\frac{\pi}{2})^2 t}$ terms go to 0, where $e^{\left(-\frac{\pi}{2}\right)^2 t}$ is the *principal* rate of decay. In general, this principal rate of decay is the largest in the matched Fourier series. So, as the length *expands* the principal rate decreases, and vice-versa.

Example 3.2.2. Consider the angular case of the above:

$$H(\theta,t) \triangleq h(r\theta,t)$$

We derive the analogous governing conditions as follows:

$$H_{\theta\theta} = r^2 h_{\theta\theta}(r\theta,t)$$
$$H_t(\theta,t) = h_t(r\theta,t)$$

Since

$$h_t(r\theta,t) = r^2 h_{\theta\theta}(r\theta,t)$$

we have that:

$$H_t(\theta,t) = \frac{1}{r^2} H_{\theta\theta}(\theta,t)$$

Thus, our governing conditions are:

$$\begin{cases} H_t(\theta,t) = (1/r)^2\, H_{\theta\theta}(\theta,t) \\ H(\theta,0) = h(r\theta,0) = g(r\theta) \end{cases}$$

As usual, we guess and match a solution as follows:

$$T(t) = e^{-\left(\frac{k}{r}\right)^2 t}$$

and we have:

$$H(\theta,t) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(k\theta) e^{-\left(\frac{k^2}{r^2}\right)t}$$

Example 3.2.3. Let us consider the following example:

$$\begin{cases} h_t = h_{xx}, \quad x \in [0,\pi],\ t > 0 \\ h(x,0) = \wedge(x) \\ h_t(x,t) = 0 \end{cases}$$

Recall that we have already taken the Fourier series expansion of $\wedge$ as:

$$\wedge(x) = \frac{4}{\pi}\left[\sin(x) - \frac{1}{3^2}\sin(3x) + \cdots + \frac{(-1)^n}{(2n+1)^2}\sin[(2n+1)x]\right]$$

We match the appropriate scaling factor to obtain:

$$h(x,t) = \frac{4}{\pi}\left[\sin(x)e^{-t} - \frac{1}{3^2}\sin(3x)e^{-3^2 t} + \cdots + \frac{(-1)^n}{(2n+1)^2}\sin[(2n+1)x]e^{-(2n+1)^2 t}\right]$$

Remark 3.2.3.1. If the heat equation is given as

$$h_t = \alpha^2 h_{xx}$$

then the scaling factor $T(t)$ is given as

$$T(t) = e^{-(\alpha k)^2 t}$$

3.3 Non-homogeneous B.V.P.'s for the heat equation

Example 3.3.1. Consider the *non-homogeneous* boundary value problem, representing a heat equation:

$$\begin{cases} u_t = u_{xx}, \quad x \in [0, 30],\ t > 0 \\ u(0,t) = 60,\ u(30,t) = 0 \\ u(x,0) = 60 - 2x \end{cases}$$

Observe that we cannot decompose this into a Fourier series that will work, because the endpoints are non-zero. Observe further that:

$$\frac{\partial}{\partial t} u(x,0) = \frac{\partial^2}{\partial x^2} u(x,0) = 0$$

And that $u(x,t) = 60 - 2x$ is a valid, *stationary* solution.

Example 3.3.2. Now consider a similar example, in which we have:

$$\begin{cases} u_t = u_{xx}, \quad x \in [0, 30],\ t > 0 \\ u(0,t) = 60,\ u(30,t) = 0 \\ u(x,0) = 60 - 2x + \sin\left(\frac{\pi}{30} x\right) \end{cases}$$

Here, we match the solution $u(x,0)$ term-by-term. Note that the first-order terms (e.g., 60, $-2x$) are already satisfied, such that $(60)_t = (60)_{xx} = 0$, and likewise with $-2x$. As usual, we match the sine term as follows:

$$\sin\left(\frac{\pi}{30} x\right) \Rightarrow \sin\left(\frac{\pi}{30} x\right) e^{-(\pi/30)^2 t}$$

Thus, our solution is:

$$u(x,t) = 60 - 2x + \sin\left(\frac{\pi}{30} x\right) e^{-(\pi/30)^2 t}$$

Observing that:

$$\lim_{t \to \infty} u(x,t) = 60 - 2x$$

Example 3.3.3 ($\star$)**.** Now consider the following equation:

$$\begin{cases} u_t = u_{xx}, \quad x \in (0, 25),\ t > 0 \\ u_x(0,t) = u_x(25,t) = 0 \\ u(x,0) = x \end{cases}$$

The stationary solution $u(x,t) = x$ is appealing, but does not satisfy the insulated boundary conditions, since $(x)_x = 1 \neq 0$. We want a function that makes it insulated (at $x = 0, x = 25$), such as:

$$\frac{\partial}{\partial x} \sin(\cdot) = 0, \quad \text{or} \quad \frac{\partial}{\partial x} \cos(\cdot) = 0$$

sin is a good choice, here, particularly $(\sin(k\pi x/25))$. But we're after a function which has a partial differential with respect to x equalling the sine function. As such, we choose:

$$\frac{\partial}{\partial x} \cos\left(k\frac{\pi}{25}x\right)\Big|_{x=0,\,25} = -\sin\left(k\frac{\pi}{25}x\right)\Big|_{x=0,\,25} = 0$$

We need to do a Fourier series decomposition along $\cos(k\pi x/25)$, and the solution will therefore be:

$$u(x,t) = \sum_{k=0}^{\infty} a_k \cos(kx) e^{-(k\pi/25)^2 t}$$

Then, all that's left to do is find the a_ks by integrating against the coordinates $\cos(k\pi x/25)$ on $x \in [0, 25]$:

$$\int_0^{25} x \cos\left(k\frac{\pi}{25}x\right)\, dx = \int_0^{25} \left(\sum_{k=1}^{\infty} a_k\right) \cos\left(\frac{\pi}{25}x\right)\, dx$$

and we get:

$$a_k = \frac{1}{25/2}\int_0^{25} x \cos\left(\frac{\pi}{25}kx\right)\, dx = \begin{cases} \dfrac{-100}{\pi^2(2n+1)^2} & k = 2n+1,\ n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Thus, the solution is:

$$u(x,t) = \frac{25}{2} - \frac{100}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\left[\frac{\pi}{25}(2n+1)x\right] e^{-\left(\frac{\pi}{25}(2n+1)\right)^2 t}}{(2n+1)^2}$$

3.3.1 Summary of heat diffusion problems

Closed wire A closed wire diffusion problem is given as:

$$\begin{cases} h_t = h_{\theta\theta} \\ h(\theta, 0) = f(\theta) \end{cases}$$

Where $h(\theta + 2\pi, t) = h(\theta, t)$ (i.e., it is 2π periodic), and $f(\theta + 2\pi) = f(\theta)$ (so is f). To solve, compute the Fourier series expansion of $f(\theta)$ using the Euler-Fourier formulas:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,d\theta \quad k \in \{0, 1, 2, \dots\}$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta)\,d\theta \qquad k \in \{1, 2, \dots\}$$

and match term-by-term with $T(t) = e^{-k^2 t}$. Thus, the solution is given as:

$$h(\theta, t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right) e^{-k^2 t}$$

Open wire; ends fixed at zero A problem of this kind is given as:

$$\begin{cases} h_t = h_{xx} \\ h(x, 0) = f(x) \\ h(0, t) = h(\ell, t) = 0 \end{cases}$$

We first decompose this using the modified Euler-Fourier formulas along coordinates $\sin\left(\frac{k\pi}{\ell}x\right)$:

$$b_k = \frac{1}{\ell/2}\int_0^{\ell} f(x)\sin(k\pi x/\ell)\,dx \quad k \in \{1, 2, \dots\}$$

...and the scaling factor $T(t) = e^{-(k\pi/\ell)^2 t}$. Thus the solution is given as:

$$h(x, t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right) e^{-(k\pi/\ell)^2 t}$$

Open wire; non-homogeneous boundaries A problem of this variant is given as:

$$\begin{cases} u_t = u_{xx}, \quad x \in [0, L],\ t > 0 \\ u(0, t) = T_1, \quad u(L, t) = T_2 \\ u(x, 0) = f(x) \end{cases}$$

We begin by rewriting $u = v + w$, where:

$$v(x) := (T_2 - T_1)\,\frac{x}{L} + T_1$$

and w is the solution to:

$$w := \begin{cases} w_{xx} = w_t \\ w(0, t) = w(L, t) = 0 \\ w(x, 0) = f(x) - v(x) \end{cases}$$

We solve w using the method described in the previous section. We write $u(x,t)$ as:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right) e^{-(k\pi/L)^2 t}$$

and:

$$b_k = \frac{2}{L}\int_0^L (f(x) - v(x))\sin\left(\frac{k\pi x}{L}\right) dx$$

Open wire; insulated ends A problem of this variant is given as:

$$\begin{cases} h_t = h_{xx} \\ h_x(0,t) = h_x(\ell,t) = 0 \\ h(x,0) = f(x) \end{cases}$$

We decompose this problem along the coordinates $\cos(k\pi x/\ell)$ as:

$$a_k = \frac{1}{\ell/2}\int_0^\ell f(x)\cos(k\pi x/\ell)\, dx \quad k \in \{0,1,2,\ldots\}$$

...and the scaling factor $T(t) = e^{-(k\pi/\ell)^2 t}$. Thus the solution is given as:

$$h(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right) e^{-(k\pi/\ell)^2 t}$$

3.4 The Wave Equation

Example 3.4.1. Consider the wave equation:

$$\begin{cases} u_{tt} = 4u_{xx}, \quad x \in (0,30),\ t > 0 \\ u_t(x,0) = \sin\left(\frac{\pi}{30}x\right) \\ u(x,0) = \sin\left(\frac{\pi}{30}x\right) + 6\sin\left(\frac{\pi}{15}x\right) \end{cases}$$

By superposition (since this equation is a homogeneous one), we separate into

$$u = v + w$$

where

$$v := \begin{cases} v_{tt} = 4v_{xx}, \quad x \in (0,30),\ t > 0 \\ v_t(x,0) = 0 \\ v(x,0) = \sin\left(\frac{\pi}{30}x\right) + 6\sin\left(\frac{\pi}{15}x\right) \end{cases} \quad \text{and} \quad w := \begin{cases} w_{tt} = 4w_{xx}, \quad x \in (0,30),\ t > 0 \\ w_t(x,0) = \sin\left(\frac{\pi}{30}x\right) \\ w(x,0) = 0 \end{cases}$$

and guess/match the solution to each of v, w term-by-term. Consider first v. We match as follows:

$$\sin\left(\frac{\pi}{30}x\right) \Rightarrow \sin\left(\frac{\pi}{30}x\right)\cos\left(\frac{2\pi}{30}t\right)$$
$$\sin\left(\frac{\pi}{15}x\right) \Rightarrow \sin\left(\frac{\pi}{15}x\right)\cos\left(\frac{2\pi}{15}t\right)$$

Note that this is a sensible matching, because it holds for $v_{tt} = 4v_{xx}$, and we can verify this as follows. Call

$$v(x,0) := \sin\left(\frac{\pi}{30}x\right)\cos\left(\frac{2\pi}{30}t\right) + \sin\left(\frac{\pi}{15}x\right)\cos\left(\frac{2\pi}{15}t\right)$$

And observe that:

$$v_{tt}(x,0) = -\left(\frac{2\pi}{30}\right)^2 \sin\left(\frac{\pi}{30}x\right)\cos\left(\frac{2\pi}{30}t\right) - \left(\frac{2\pi}{15}\right)^2 \sin\left(\frac{\pi}{15}x\right)\cos\left(\frac{2\pi}{15}t\right)$$

as well as:

$$\begin{aligned} v_{xx}(x,0) &= -\left(\frac{\pi}{30}\right)^2 \sin\left(\frac{\pi}{30}x\right)\cos\left(\frac{2\pi}{30}t\right) - \left(\frac{\pi}{15}\right)^2 \sin\left(\frac{\pi}{15}x\right)\cos\left(\frac{2\pi}{15}t\right) \\ &= \frac{1}{4}v_{tt}(x,0) \end{aligned}$$

We continue for w, and guess/match that:

$$\sin\left(\frac{\pi}{30}x\right) \Rightarrow \sin\left(\frac{\pi}{30}x\right)\frac{\sin\left(\frac{2\pi}{30}t\right)}{2\pi/30}$$

so:

$$w(x,t) = \sin\left(\frac{\pi}{30}x\right)\frac{\sin\left(\frac{2\pi}{30}t\right)}{2\pi/30}$$

and that:

$$\begin{aligned} u(x,t) &= v + w \\ &= \left[\sin\left(\frac{\pi}{30}x\right)\cos\left(\frac{2\pi}{30}t\right) + \sin\left(\frac{\pi}{15}x\right)\cos\left(\frac{2\pi}{15}t\right)\right] + \left[\sin\left(\frac{\pi}{30}x\right)\frac{\sin\left(\frac{2\pi}{30}t\right)}{2\pi/30}\right] \end{aligned}$$

Example 3.4.2 ($\star$). Now, we consider a more general example of the above:

$$\begin{cases} u_{tt} = 4u_{xx}, & x \in (0,30),\ t > 0 \\ u_t(x,0) = g(x), & x \in (0,30) \\ u(x,0) = f(x), & x \in (0,30) \end{cases}$$

To solve, we rewrite as $u = v + w$, and decompose $f(x)$, $g(x)$ by a sine-Fourier expansion along $(0,30)$. We begin with v:

$$\begin{cases} v_{tt} = 4v_{xx}, & x \in (0,30),\ t > 0 \\ v_t(x,0) = 0 \\ v(x,0) = f(x) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{\pi}{30}kx\right), & x \in (0,30) \end{cases} \qquad c_k = \frac{1}{30/2}\int_0^{30} f(x)\sin\left(\frac{\pi}{30}kx\right)\,dx$$

similarly, with w:

$$\begin{cases} w_{tt} = 4w_{xx}, \quad x \in (0,30),\, t > 0 \\ w_t(x,0) = g(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi}{30}kx\right), \quad x \in (0,30) \quad b_k = \frac{1}{30/2}\int_0^{30} g(x)\sin\left(\frac{\pi}{30}kx\right)\, dx \\ w(x,0) = 0 \end{cases}$$

Then we guess and match v, w individually. By superposition, we get:

$$u(x,t) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{\pi}{30}kx\right)\cos\left(2\frac{\pi}{30}kt\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi}{30}kx\right)\frac{\sin\left(2\frac{\pi}{30}kt\right)}{k\pi/15}$$

3.5 The Laplace Equation

In $\mathbb{R}^3$, the Laplacian operator $\triangle$ is defined as:

$$\triangle u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u$$

Some examples of solutions include:

1-d $ax + b$

2-d x, y, $x^2 - y^2$, xy, $x^3 - 3xy^2$, $\text{Re} \mid \text{Im}(x+iy)^k$, $\text{Re} \mid \text{Im}(re^{i\theta})^k$, $\ln(r)$, and so on.

3-d xyz, $(x^2 - y^2)z$, $\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

Example 3.5.1. One straightforward case is solving the Laplace equation in 1-d.

$$\begin{cases} \triangle u = u_{xx} = 0 \\ u_x(0) = a \\ u(0) = b \end{cases} \implies u(x) = ax + b$$

and:

$$\begin{cases} \triangle u = u_{xx} = 0 \\ u(0) = c \\ u(1) = d \end{cases} \implies u(x) = \frac{d-c}{1}x + c$$

Example 3.5.2 ($\star$). Consider a square ($[0,\pi]^2$) in 2-d, with the following: $u(x,0) = u(x,\pi) = u(0,y) = 0$, $u(\pi,y) = f(y)$, and $\triangle u = 0$. For the sake of example, we consider the case $f(y) = \sin(y)$.

We guess that $u(x,y) = \sin(y)X(x)$. We then consider $\triangle u$ as:

$$\begin{aligned} \triangle u &= u_{xx} + u_{yy} \\ &= \sin(y)X''(x) + -\sin(y)X(x) \\ &= \sin(y)(X''(x) - X(x)) \end{aligned}$$

Or that X satisfies the equation:

$$X''(x) - X(x) = 0, \quad \forall x.$$

Which leaves us with the options $X = e^x$, or $X = e^{-x}$. Thus, the general solution is:

$$u(x, y) = c_1 e^x \sin(y) + c_2 e^{-x} \sin(y)$$

Now we apply the boundary conditions. Begin with the left $x = 0$:

$$(c_1 + c_2)\sin(y) = 0 \Longrightarrow c_1 = -c_2 = c$$

and the right $x = \pi$:

$$c(e^{\pi} - e^{-\pi})\sin(y) = \sin(y) \Longrightarrow c = \frac{1}{e^{\pi} - e^{-\pi}}$$

So, when $f(y) = \sin(y)$, we have:

$$u(x, y) = \frac{1}{e^{\pi} - e^{-\pi}}(e^x - e^{-x})\sin(y)$$

Likewise, when $f(y) = \sin(ky)$, we have:

$$u(x, y) = \frac{1}{e^{k\pi} - e^{-k\pi}}(e^{kx} - e^{-kx})\sin(y)$$

and in general, when $f(y)$ is not just a $\sin(ky)$, we decompose it in terms of $\sin(ky)$ as:

$$f(y) = \sum_{k=1}^{\infty} b_k \sin(ky), \quad b_k = \frac{1}{\pi/2}\int_0^{\pi} f(y)\sin(ky)\,dy$$

and the solution is:

$$u(x, y) = \sum_{k=1}^{\infty} \frac{b_k}{e^{k\pi} - e^{-k\pi}}(e^{kx} - e^{-kx})\sin(y)$$

Example 3.5.3. Now consider a rectangle $[0, a] \times [0, \pi]$. We take the sine-Fourier decomposition of $f(y)$ on $[0, \pi]$. The solution is given as:

$$u(x, y) = \sum_{k=1}^{\infty} \frac{b_k}{e^{ka} - e^{-ka}}(e^{kx} - e^{-kx})\sin(y)$$

Example 3.5.4. Now, consider an *arbitrary* rectangle on $[0, a] \times [0, b]$. We first decompose $f(y)$ into a sine-Fourier series over $\sin\left(\frac{\pi}{b}ky\right)$ on $[0, b]$, with:

$$f(y) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi}{b}ky\right), \quad b_k = \frac{1}{b/2}\int_0^{b} f(y)\sin\left(\frac{\pi}{b}ky\right)\,dy$$

and match terms as:

$$u(x, y) = \sum_{k=1}^{\infty} \frac{b_k}{e^{\pi ka/b} - e^{-\pi ka/b}}\left(e^{\frac{\pi}{b}kx} - e^{-\frac{\pi}{b}kx}\right)\sin\left(\frac{\pi}{b}ky\right)$$

Example 3.5.5 ($\star$)**.** In general, consider a rectangle on $[0, K] \times [0, L]$, which has boundary values, $f_1(x), f_2(x), g_1(y), g_2(y)$ on the bottom, top, left, and right, respectively. By superposition, we can solve this by decomposing the solution into 4 homogeneous subcomponents:

$$u = u_L + u_R + u_T + u_B$$

The solution is given as:[2]

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{L}\right) \frac{\sinh\left(\frac{n\pi}{L}(K-x)\right)}{\sinh\left(\frac{n\pi}{L}K\right)} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{L}\right) \frac{\sinh\left(\frac{n\pi}{L}x\right)}{\sinh\left(\frac{n\pi}{L}K\right)}$$
$$+ \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{K}\right) \frac{\sinh\left(\frac{n\pi}{K}(L-y)\right)}{\sinh\left(\frac{n\pi}{K}L\right)} + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{K}\right) \frac{\sinh\left(\frac{n\pi}{K}y\right)}{\sinh\left(\frac{n\pi}{K}L\right)}$$

Where:

$$a_n = \frac{1}{2/L} \int_0^L g_1(y) \sin\left(\frac{n\pi}{L}y\right) dy$$
$$b_n = \frac{1}{2/L} \int_0^L g_2(y) \sin\left(\frac{n\pi}{L}y\right) dy$$
$$c_n = \frac{1}{2/K} \int_0^K f_1(y) \sin\left(\frac{n\pi}{K}x\right) dx$$
$$d_n = \frac{1}{2/K} \int_0^K f_2(y) \sin\left(\frac{n\pi}{K}x\right) dx$$

Example 3.5.6 ($\star$)**.** Now, we'll solve the Laplace equation on a disk, where:

$$\begin{cases} \triangle u = u_{xx} + u_{yy} \\ \qquad = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ \qquad = 0 \\ u(1,\theta) = f(\theta),\ f(\theta + 2\pi) = f(\theta) \end{cases}$$

Let's first consider the example where $f(\theta) = sin(k\theta)$. We guess that $u(r,\theta) = \sin(k\theta)R(r)$. Then:

$$\triangle u = \left(R'' + \frac{1}{r}R'\right)\sin(k\theta) + \frac{R}{r^2}(-k^2\sin(k\theta))$$

and we conclude that a sufficient correcting factor R must have:

$$R'' + \frac{1}{r}R' - \frac{k^2}{r^2}R = 0$$

so we guess that $R(r) = r^\alpha$:

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - k^2 r^{\alpha-2} = 0$$

[2]For a more detailed exposition, see: `http://faculty.wwu.edu/curgus/courses/math_pages/math_430/Laplace_equation_rectangle.html`.

or:

$$[\alpha(\alpha-1)+\alpha-k^2]r^\alpha=0$$

meaning that $\alpha=\pm k$. Here, we must choose $k>0$, so that the solution is bounded. Thus, we have:

$$u(r,\theta)=r^k\sin(\theta)=\text{Im}\left(\sum_{n=1}^{k}\binom{k}{n}x^ny^{k-n}\right)$$

Similarly, when we have $f(\theta)=\cos(k\theta)$, the solution is:

$$u(r,\theta)=r^k\cos(\theta)$$

And in general, when we have $f(\theta)$, we can take the Fourier series expansion:

$$f(\theta)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(k\theta)+b_k\sin(k\theta)$$

where:

$$a_k=\frac{1}{2\pi/2}\int_0^{2\pi}f(\theta)\cos(k\theta)\,d\theta,$$
$$b_k=\frac{1}{2\pi/2}\int_0^{2\pi}f(\theta)\sin(k\theta)\,d\theta,$$

to obtain the solution:

$$u(r,\theta)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right)r^k$$