MATH 324 D, Braune

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Contents

Chapter 15

Multiple Integrals

15.1 Double Integrals over Rectangles

Recall that we approximate an integral of a function of a single variable by subdividing the region $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ with $\Delta x = (b-a)/n$, and approximate the integral as $n \to \infty$ as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We can interpret the above Riemann summation over a single variable as analogous to determining the *area* below f, where f is a positive function. Likewise, we can take Riemann integrals over *areas* to determine volume.

If we let the region R be divided into $m \times n$ sub-rectangles, and define the area of each to be $\Delta A = \Delta x \Delta y$ (where $\Delta x = (b - a/n)$, and Δy is defined analogously), then we have that:

$$V \approx f(x_{ij}^*, y_{ij}^*)\Delta A$$

and that:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

The quantity above stated on either side of the equality is the volume of the surface above the region R and below the surface f.

15.1.1 Midpoint Rule

One can approximate the value of a double integral by taking a pseudo-Riemann sum evaluated only at the midpoints of each sub-region:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_{ij}, \overline{y}_{ij})\Delta A$$

15.1.2 Average Value

Recall that we had earlier for the integral over a function of one variable that:

$$\overline{f}_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Similarly, in the two-variable case, we have that:

$$\overline{f}_R = \frac{1}{A(R)}\iint_R f(x)\,dx$$

15.1.3 Properties of Double Integrals

Finally, note that double integrals are *linear*, meaning that they exhibit the following properties:

$$\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

Likewise, when c is a constant, we have that:

$$\iint_R c\,f(x,y)\,dA = c\iint_R c\,f(x,y)\,dA$$

Finally, if $f \geq g$ for all $(x,y) \in R$, then:

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

15.1.4 Iterated Integrals

Consider $R = [a,b]\times[c,d]$, and observe that:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

When evaluating this integral, we:

- Work from the inside out, integrating f first with respect to a change in y, then with respect to a change in x.
- "Hold" the variables not being integrated as *constant* with respect to the variable currently being integrated against.

Example 15.1.1.

$$\begin{aligned}\int_0^3\int_1^2 x^2y\,dy\,dx &= \int_0^3 \left[x^2\frac{y^2}{2}\right]_{y=1}^{2} dx \\ &= \int_0^3 x^2\left[\frac{2^2-1^2}{2}\right]dx = \frac{3}{2}\int_0^3 x^2\,dx \\ &= \frac{3}{2}\left[\frac{x^3}{3}\right]_0^3 = \boxed{\frac{27}{2}}\end{aligned}$$

Note that it is often convenient to change the *order* in which a function is integrated, and that this is a legal operation due to a theorem of Fubini.

Theorem 15.1.1 (Fubini's)**.** If f is continuous on a rectangle $R = [a, b] \times [c, d]$, then:

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

15.2 Double Integrals over General Regions

Call $D \subsetneq \mathbb{R}^2$ a non-rectangular region over which we would like to integrate $f(x,y)$. Then, we define:

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

if F is integrable over R (any rectangular region enclosing D), then we have that:

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

- Call D a *type I* region if D is defined:

  $$D = \{(x,y) : x \in [a,b], y \in [g_1(x), g_2(x)]\}$$

  where it is convenient to treat the nested integral over a single x_i, thus lending a natural form:

  $$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

- Likewise, we call D a *type II* region if DA is defined

  $$D = \{(x,y) : y \in [c,d], x \in [h_1(y), h_2(y)]\}$$

  and it is likewise convenient to integrate it as

  $$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

- If D is more complicated, then we subdivide it into regions D_i such that each D_i is itself either a type-I or type-II region and:

  $$D = \bigcup_i D_i, \quad \emptyset = D_i \cap D_j,\ i \neq j$$

  where the later statement means that any pairwise intersection of the sub-divided regions is non-overlapping. Therefore:

  $$\iint_D f(x,y)\, dA = \sum_i \int_{D_i} f(x,y)\, dA$$

Note the following two properties of double integrals: first, a constant dA integrated over D is the area of that region D:

$$\iint_D dA = A(D)$$

and that if $f(x,y) \in [m, M]$ for all $(x,y) \in D$, then:

$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$

15.3 Double Integrals in Polar Coordinates

Recall the polar coordinate system, which often makes it easier to integrate over circular or other complicated regions. Specifically, recall that:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \theta = \arctan(y/x)$$

a *polar rectangle* is thusly defined as:

$$R = \{(r,\theta) : r \in [a,b],\, \theta \in [\alpha,\beta]\}$$

and note that:

$$dA = r\,dr\,d\theta$$

15.4 Applications of Double Integrals

- To find the mass M of a planar object with density function $\rho : \mathbb{R}^2 \to \mathbb{R}$, note that:

$$M = \iint_D \rho(x,y)\,dA$$

- To find the average value of f in D, we have that:

$$\overline{f} = \frac{1}{A(D)} \iint_D f(x,y)\,dA$$

  and similarly to find the weighted average (according to the corresponding density function ρ), we have:

$$\overline{f} = \frac{1}{M(D)} \iint_D \rho(x,y) f(x,y)\,dA$$

- We also have that, for a planar object with density ρ, that the center of mass $(\overline{x}, \overline{y})$ is given as:

$$\overline{x} = \frac{1}{M(D)} \iint_D x\rho(x,y)\,dA, \quad \overline{y} = \frac{1}{M(D)} \iint_D y\rho(x,y)\,dA$$

- Lastly, we can also compute the *moment of inertia* of an object spinning about some axis of rotation. Note that $\Delta m = \rho \Delta A$, and that $v = \omega r$. Then, we have that:

$$\begin{aligned} KE &= \frac{1}{2}(\Delta m)v^2 \\ &= \frac{1}{2}(\Delta m)(\omega r)^2 \\ &= \frac{1}{2}(\rho \Delta A)(\omega r)^2 = \frac{1}{2}(r^2 \rho \Delta A)\omega^2 \end{aligned}$$

and that the moment of inertia about the origin is:

$$I_0 = \iint_D r^2 \rho \, dA$$

Likewise, the moment of inertia about the x-axis is:

$$I_x = \iint_D y^2 \rho \, dA$$

and finally, about the y-axis:

$$I_y = \iint_D x^2 \rho \, dA$$

15.5 Surface Area

Let S be the surface area of a solid with equation $z = f(x, y)$, where f has continuous partial derivatives with respect to the x-, y- and z-axes.

For simplicity, we assume that $f(x, y) \geq 0$, and that the domain D is over a rectangle. We use a Riemann-like process and divide D into sub-rectangle R_{ij}, and a corresponding point $P_{ij}(x_i, y_j, f(x_i, y_j))$ where $(x_i, y_j$ is the point in the region closest to the origin.

Taking the summation over all tangent planes, we arrive at:

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

where T_{ij} is the area of the tangent plane.

We must now find the area of some ΔT for a given i, j pairing. For small enough change in x, y, this portion of the surface represents a parallelogram in 3-dimensional space. It is uniquely defined by two perpendicular curves, which at infinitesimal zoom appear as vectors, not curves. Let us call these vectors $\vec{a}$ and $\vec{b}$. Note that:

$$\begin{aligned} \vec{a} &= (\Delta x, 0, f(x_0 + \Delta x, y_0) - f(x_0, y_0)) \\ &= \left(\Delta x, 0, \frac{\partial f}{\partial x}(x_0, y_0)\Delta x\right) \end{aligned}$$

$$\begin{aligned} \vec{b} &= (0, \Delta y, 0, f(x_0, y_0 + \Delta y) - f(x_0, y_0)) \\ &= \left(0, \Delta y, \frac{\partial f}{\partial y}(x_0, y_0)\Delta y\right) \end{aligned}$$

To compute the area of this parallelogram, we take the matrix determinant:

$$\begin{aligned} a \times b &\approx \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x & 0 & f_x(x,y)\Delta x \\ 0 & \Delta y & f_y(x,y)\Delta y \end{vmatrix} \\ &= (-f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y) \end{aligned}$$

Then, the magnitude of $|\Delta T| = |\vec{a} \times \vec{b}|$:

$$\begin{aligned} |\Delta T| &\approx |\vec{a} \times \vec{b}| \\ &= \sqrt{f_x^2 + f_y^2 + 1}\, \Delta x \Delta y \end{aligned}$$

Integrating over the whole suface, we have:

$$\begin{aligned} A(S) &= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA \\ &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \end{aligned}$$

15.6 Triple Integrals

Whereas in the double integral case we integrated over a rectangular region (in the simple case), in the tripe integral case (by contrast) we integrate over $D \subsetneq \mathbb{R}^3$. A simple case is $D = [a, b] \times [c, d] \times [e, f]$.

Again, a Riemann-like argument shows that:

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{n=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) = \iiint_D f(x, y, z)\, dV$$

Note also that Fubini's theorem extends over triple integrals as well, meaning that dV can be broken down into any of the six combinations of $dx\, dy\, dz$.

We define three types of solid regions used to classify and evaluate triple integrals:

- A *type I* integral is given as $E = \{(x, y, z) : (x, y) \in D, z \in [u_1(x, y), u_2(x, y)]\}$ (that is, D is some region, and z varies between two heights.)

  In this instance, it is often convenient to evaluate it as:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

- A *type II* integral is given when x varies between two constants, but y varies as a function of x, and z as a function of x and y. In this instance, it is convenient to integrate it the region as:

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

- Finally, a *type III* region is one where $(x,z) \in D$, and y varies as a function of x and z. It is typically convenient to integrate integrals of this form as:

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

15.6.1 Applications of Triple Integrals

- To find the *mass* occupied by some volume, we integrate the density of that space with respect to an infinitesimal change in volume:

$$m = \iiint_R \rho(x,y,z)\,dV$$

- Likewise, we can find the *moments* about the three coordinate axes as follows:

$$M_{yz} = \iiint_R x\rho(x,y,z)\,dV \qquad M_{xz} = \iiint_R y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_R z\rho(x,y,z)\,dV$$

- The above quantities can be used to determine the center of mass, which is given as: $(\overline{x}, \overline{y}, \overline{z})$:

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}$$

- Finally, if $\rho = c$ for all $(x,y,z) \in R$, then the center of mass is called the *centroid*. Likewise, the three moments of inertia about the coordinate axes are:

$$I_x = \iiint_R (y^2+z^2)\rho(x,y,z)\,dV \qquad I_y = \iiint_R (x^2+z^2)\rho(x,y,z)\,dV$$

$$I_z = \iiint_R (x^2+y^2)\rho(x,y,z)\,dV$$

15.7 Triple Integrals in Cylindrical Coordinates

In a cylindrical coordinate system, we make the following translation:

$$(x,y,z) \rightsquigarrow (r\cos\theta, r\sin\theta, z)$$

where:

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

Say we have a region E (of type I) which has projection D such that:

$$E = \{(x,y,z) : (x,y) \in D,\ z \in [u_1(x,y), u_2(x,y)]\}$$

and:

$$D = \{(r, \theta) : \theta \in [\alpha, \beta],\, r \in [h_1(\theta), h_2(\theta)]\}$$

We have that:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

or, converting to cylindrical coordinates, that:

$$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

15.8 Triple Integrals in Spherical Coordinates

Another useful coordinate system is the *spherical coordinate system*, which represents points (ρ, θ, ϕ), where $\rho = ||P||_2$, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the vector $\vec{P}$.
Note that:

- $\rho = c$ is a sphere with radius c.
- $\theta = c$ is the half-plane with angle c about the positive x-axis.
- $\phi = c$ is a half cone which is either above or below the xy-plane, depending on whether ϕ is less than or greater than $\pi/2$, respectively.

To convert between spherical and rectangular coordinates, observe the following relationships:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi$$

and that:

$$\rho^2 = x^2 + y^2 + z^2$$

Our "rectangular" unit in spherical coordinates is the spherical *wedge*, which is given as:

$$E = \{(\rho, \theta, \phi) : \rho \in [a, b],\, \theta \in [\alpha, \beta],\, \phi \in [c, d]\}$$

We divide E into smaller spherical wedges, E_{ijk}, by equally spaced spheres, half-planes, and half-cones. The volume is approximately a rectangular box, with $\Delta\rho$, $\rho_i \Delta\phi$, and $\rho_i \sin\phi_k \Delta\theta$. Thus, the approximate volume of E_{ijk} is given as:

$$\begin{aligned} \Delta V_{ijk} &\approx (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin\phi_k \Delta\theta) \\ &= \rho_i^2 \sin\phi_k \Delta\rho \Delta\theta \Delta\phi \end{aligned}$$

An application of the Mean Value Theorem yields exact equality on ΔV_{ijk}, or that:

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho \Delta\theta \Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k) \in E_{ijk}$. Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of this point. Then:

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

Hence,

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\rho_i^2 \sin\phi_k\,d\rho\,d\theta,d\phi$$

15.9 Change of Variables in Multiple Integrals

Recall that in single-variable calculus, we often perform a change of variable on a 1-dimensional integral as follows:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,\frac{dx}{du}\,du$$

More generally, we consider a transformation T from the uv-plane to the xy-plane as:

$$T(u,v) = (x,y)$$

where $x = g(u,v)$, and $y = h(u,v)$ (sometimes this is written as: $x = x(u,v)$, and $y = y(u,v)$). Note that we assume T is a C^1 *transformation*, meaning that g and h have continuous first-order partial derivatives. Note that the inverse of these transformations are denoted $u = G(x,y)$ and $v = H(x,y)$, respectively.
Say we have the image of S (some region in uv-space) projected onto the xy-plane. Note that we can write a function for the location of the position vector in the image as follows:

$$\vec{r}(u,v) = \langle g(u,v),\, h(u,v)\rangle$$

Note also that we can take the tangent vectors as the first-order derivatives of $\vec{r}$ as:

$$\vec{r}_u = \langle g_u(u_0,v_0),\, h_u(u_0,v_0)\rangle$$

and similarly:

$$\vec{r}_v = \langle g_v(u_0,v_0),\, h_v(u_0,v_0)\rangle$$

Then, we have a parallelogram whose edges are defined by $\vec{a}, \vec{b}$ as:

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0,v_0)$$
$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0,v_0)$$

But, we have that:

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u}$$

so:

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \vec{r}_u$$

and:

$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \vec{r}_v$$

This means that we can take the Jacobian to determine the area ΔA. The transformation *Jacobian* is given as:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = [(\partial x/\partial u)(\partial y/\partial v)] - [(\partial x/\partial v)(\partial y/\partial u)]$$

Thus we have (from the previous calculation):

$$\Delta A \approx \left| \frac{\delta(x,y)}{\delta(u,v)} \right| \Delta u \, \Delta v$$

Note that:

$$\begin{aligned} \iint_D f(x,y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v \\ &= \iint_R f(g(u,v),\, h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \end{aligned}$$

In integrals of three variables, the situation is the same. Let T be a transformation map from S in uvw-space to R in xyz-space by:

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

Then, the Jacobian of T is given as:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

And thus that:

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w),\, y(u,v,w),\, z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

Chapter 14

Partial Derivatives

14.5 The Chain Rule

Recall the Chain Rule for single variable functions gives that if $y = f(x)$ and $x = g(t)$, that y is indirectly differentiable by a change in t, and that:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

There are several such cases of the Chain Rule, each described here. First, we deal with the case where $z = f(x, y)$, and both x and y are differentiable by t. That is, if $z = f(g(t), h(t))$, then:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second case, we deal with $z = f(x, y)$ where x and y are each a function of two variables, say $x = g(s, t)$, $y = h(s, t)$. Then, we have that:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

We now present the general case of the Chain Rule. Suppose that u is a differentiable function of $x_1, x_2, \ldots, x_n$, and that each x_j (for $j \in [n]$) was itself differentiable by $t_1, t_2, \ldots, t_m$. Then, u is a function of $t_1, t_2, \ldots, t_m$, and:

$$\begin{aligned}\frac{\partial u}{\partial t_i} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \\ &= \sum_{j \in [n]} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}, \quad \text{for } i \in [m].\end{aligned}$$

14.6 Directional Derivatives and the Gradient Vector

Suppose now that we have a function f of two variables such that $z = f(x, y)$. We can observe the *gradient* of f, which is the vector along which z is tangent at some (x, y), by the following:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

Intuitively, ∇f gives us the direction along which f *increases* the fastest, whereas $|\nabla f|$ gives us the fastest *rate* of increase.

14.6.1 Directional Derivatives

Suppose that instead of taking the rate of change along the basis vectors in any particular subspace, that instead you wanted to compute the rate of change along some arbitrary vector $\vec{u}$, instead. Then, we proceed with the following derivation:

For some $u = \langle a, b \rangle$, we have that $\vec{r}(t) = \langle a_0 + at, b_0 + bt \rangle$. Then, the directional derivative is:

$$D_{\hat{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb)}{h}$$
$$= \left. \frac{df(\vec{r}(t))}{dt} \right|_{t=0}$$
$$= \nabla f \cdot \hat{u}$$

That is, that $D_{\hat{u}} f$ is tangent to the slope along of f cut through the vertical plane containing $\hat{u}$. Note that:

$$D_{\hat{u}} f = \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Note that this obeys the same max/min rules as the standard dot product, and that $D_{\hat{u}} f$:

- ...is maximized when ∇f is in the direction of $\hat{u}$ (when $\theta = 0$).
- ...is minimized under the opposite conditions (when $\theta = \pi$).
- ...is zero when $\nabla f \perp \hat{u}$ (at $\theta = \pi/2$).

Chapter 16

Vector Calculus

16.1 Vector Fields

A vector field is a space over $\mathbb{R}^n$ whose range is $\mathbb{V}_n$ (that is, each "point" takes on the vector of a value in the same number of dimension).

We describe such a field F as the component-wise sum of component functions P, Q, which is to say that:

$$\begin{aligned}\vec{F}(x,y) &= P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} \\ &= \langle P(x,y), Q(x,y)\rangle = P\hat{\imath} + Q\hat{\jmath}\end{aligned}$$

Note that we can also describe another kind of vector field over some subset of $\mathbb{R}^3$; that is the *gradient field* of some vector-valued function $\vec{f}(x,y,z)$, which is:

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

16.2 Line Integrals

16.2.1 Parameterizations

First, say that $\vec{r} : [a,b] \to \mathbb{R}^2$ is a *parameterization* of $\mathcal{C}$ iff $\vec{r}(t) = \langle x(t), y(t)\rangle$ is in $\mathcal{C}$ for all $t \in [a,b]$, and that it traverses $\mathcal{C}$ once. Some examples follow:

1. A half-circle is parameterized as follows. For $\vec{r}(t) = \langle \cos t, \sin t\rangle$, and $t \in [0,\pi]$:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

2. An ellipse of the equation $(x-5)^2/4 + (y-3)^2 = 1$ is parameterized for $t \in [0, 2\pi]$ as follows:

$$\begin{cases} x = 5 + 2\cos t \\ y = 3 + \sin t \end{cases}$$

16.2.2 Line Integrals

Say that we have a curve $\mathcal{C}$ parameterized as $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$ for $t \in [a, b]$. Let us then divide the parameter into n sub-intervals of equal width as usual, and corresponding points $P_i(x_i, y_i)$ dividing $\mathcal{C}$ into sub-arcs Δs_i for $i \in [n]$. If we let $P_i^*(x_i^*, y_i^*)$ be any point on the ith sub-arc, then the line integral of f across $\mathcal{C}$ is:

$$\begin{aligned}\int_{\mathcal{C}} f(x,y)\,ds &= \lim_{\Delta S_i \to 0} \sum_{i\in[n]} f(x_i, y_i)\Delta S_i \\ &= \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_a^b f(\vec{r}(t))|\vec{r}'(t)|\,dt\end{aligned}$$

Example 16.2.1. Suppose that we wish to compute the area of the *fence* underneath the half-circle of radius 1 centered at the origin. Then, if $f = 2 + x^2 y$, we have a parameterization:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

and that the line integral is computed as:

$$\begin{aligned}\int_{\mathcal{C}} f(x,y)\,ds &= \int_a^b f(\cos t, \sin t)\,ds \\ &= \int_a^b 2 + \cos^2 t \sin t \underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\,dt \\ &= \int_a^b 2 + \cos^2 t \sin t dt = 2t - \left[\frac{1}{3}\cos^3(t)\Big|_{t=0}^{\pi}\right]\end{aligned}$$

16.2.3 Applications of Line Integrals

1. To compute the length of a line $\mathcal{C}$, one has:

$$\int_{\mathcal{C}} ds = \text{length}(\mathcal{C})$$

2. To compute the average value of a function f over a line $\mathcal{C}$, we have that:

$$\bar{f} = \frac{1}{\ell}\int_{\mathcal{C}} f\,ds$$

3. If $\rho(x, y)$ is the density of some function along the fence described by f and $\mathcal{C}$, then:

$$M = \int_{\mathcal{C}} \rho\,ds$$

and:

$$\overline{x} = \frac{1}{M}\int_{\mathcal{C}} x\rho\, ds$$

furthermore:

$$I_y = \frac{1}{M}\int_{\mathcal{C}} (x^2+z^2)\rho\, ds$$

Note that we can use this definition to compute the *work* done by moving some particle along a path $\mathcal{C}$ parameterized by $\vec{F} = \langle P(x,yz), Q(x,y,z), R(x,y,z)\rangle$ by considering the ith sub-arc of $\mathcal{C}$ and the work done to move across that:

$$\vec{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \vec{T}(t_i^*)]$$

or that the total work done is approximately:

$$\sum_{i\in[n]} [\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)]\Delta s_i$$

and that:

$$\lim_{n\to\infty} \sum_{i\in[n]} [\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)]\Delta s_i = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\, ds$$

where $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

16.3 The Fundamental Theorem for Line Integrals

Before discussing the Fundamental Theorem for Line Integrals, let us first discuss a couple of kinds of line integrals:

1. If we have that $F = \langle P, Q\rangle$ (or, alternatively, that $F = \langle P, Q, R\rangle$ if we are in $\mathbb{R}^3$), then we can write that:

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_{\mathcal{C}} \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle \\ &= \int_{\mathcal{C}} P\, dx + Q\, dy + R\, dz\end{aligned}$$

   where P, Q, and R take $\mathbb{R}^3 \to \mathbb{R}$ (i.e., that they are a function of (x,y,z)), and that dx, dy, and dz are the residual differentials (when with respect to time) of a parameterization of $\mathcal{C}$.

   Alternatively, we can write this same kind of integral as:

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{r(t)\in\mathcal{C}} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

2. Secondly, we discuss line integrals over vector fields:

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = \int_{\vec{r}(t)\in\mathcal{C}} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$
$$= \int_{\vec{r}(t)\in\mathcal{C}} \left[\frac{d}{dt} f(\vec{r}(t))\right] dt$$

Theorem 16.3.1. Let $\mathcal{C}$ by a smooth curve parameterized by $\vec{r}(t)$ for $t \in [a,b]$. Let f be a differentiable function of two or three variables, whose gradient vector ∇f is continuous on $\mathcal{C}$. Then:

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Proof.

$$\begin{aligned}\int_{\mathcal{C}} \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a)).\end{aligned}$$

□

Note that this implies the following three equivalent properties:

1. $\vec{F}$ is path-independent (i.e., that if any other paths have the same endpoints that they integrate the same work).

2. $\vec{F}$ is conservative (i.e., that if $\mathcal{C}$ is a closed loop that $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$).

3. $\vec{F}$ is a gradient field (i.e., that $\vec{F} = \nabla f$.

One way to visualize that (1) $\iff$ (2) is that one can form a closed loop $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same endpoints and flow in the same direction. Note that this implies path-independence from a conservative path (i.e., that $\mathcal{C}$ is conservative).

16.3.1 Tests for Gradient Fields

Note that it is often important to test whether a field is indeed that of a gradient field for some function f. One way to test this in two dimensions is as follows:

Say that $\vec{F} = \langle P, Q\rangle = \nabla f$. Then, we have that:

$$P = \frac{\partial f}{\partial x} = f_x, \quad \text{and} \quad Q = \frac{\partial f}{\partial y} = f_y$$

therefore, if we consider P_y and Q_x, it is clear that $f_{xy} = f_{yx}$ and that this must be a gradient field. Note that F must be defined at every point in the plane for this to be the case (i.e., the gradient ∇f must be defined everywhere).

Another example follows for the three-dimensional case. Let $\vec{F} = \langle P, Q, R \rangle$. If $\vec{F} = \nabla f$, then:

$$P_y = f_{xy} = f_{yx} = Q_x$$
$$Q_z = f_{yz} = f_{zy} = R_y$$
$$R_x = f_{xz} = f_{zx} = P_z$$

Theorem 16.3.2. If $\vec{F} = \langle P, Q, R \rangle$ is defined for all $(x, y, z) \in \mathbb{R}^3$, and $P_y = Q_x$, $Q_z = R_y$, and $R_x = P_z$, then there exists some f for which $\vec{F} = \nabla f$.

Example 16.3.1. Suppose that we have:

$$f_x = y^2$$
$$f_y = 2xy + e^{3z}$$
$$f_z = 3ye^{3z}$$

Then we have that $\int f_x \, dx = f = xy^2 + g(y, z)$, which we can differentiate to obtain $f_y = 2xy + g_y(y, z)$, which we know from the given is equal to $2xy + e^{3z}$, implying that $g_y(y, z) = e^{3z}$. Then we have that $g(y, z) = ye^{3z} + h(z)$, so that $f = xy^2 + ye^{3z} + h(z)$, and that after a similar such computation that $h = K$.

So, we conclude that:

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

and that we are in a vector field, or equivalently that $\nabla f = \vec{F}$.

16.4 Green's Theorem

Theorem 16.4.1 (Green's)**.**

$$\oint_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

We use $\oint_C$ to indicate that C is a bounded line with positive orientation.

Consider the following example:

Example 16.4.1. Say we wish to integrate the work done by $\vec{F} = \langle 3y - e^{\sin x}, 7x + \sqrt{y^4 + 1} \rangle$ along the path C described by the positively-oriented rotation about the origin of $r^2 = 9$.

Then:

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \oint_C \left(3y - e^{\sin x}\right)\,dx + \left(7x + \sqrt{y^4+1}\right)\,dy \\
&= \iint_D \left(\frac{\partial}{\partial x}\left[7x + \sqrt{y^4+1}\right] - \frac{\partial}{\partial y}\left[3y - e^{\sin x}\right]\right) dA \\
&= \int_0^{2\pi}\int_0^3 (7-3)\, r\, dr\, d\theta \\
&= 36\pi
\end{aligned}$$

Note that for Green's Theorem to be applicable, we must have that the curve $\mathcal{C}$ is *closed*, i.e., that it has no "endpoint". Likewise, we must have that the curve is *positively oriented*, meaning that the region D is bounded by the left-hand side of the curve when travelling along it.

16.4.1 Criterion for Gradient Fields

Note also that when applying Green's theorem, if $\vec{F} = \nabla\vec{f}$, then any closed line integral will equal 0. This is trivial from the fact that $\vec{F} = \nabla\vec{f} \implies Q_x = P_y$, and:

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \oint_C P\,dx + Q\,dy \\
&= \iint_D (Q_x - P_y)\,dA = 0
\end{aligned}$$

16.4.2 Green's Theorem for Area

To find the area enclosed by a positively oriented $\mathcal{C}$, we can choose a field $\vec{F} = \langle P, Q\rangle$ such that $Q_x - P_y = 1$, and then we have that:

$$\oint_C P\,dx + Q\,dy = \iint_D 1\,dA$$

16.5 Curl and Divergence

For a vector field $\vec{F} = \langle P, Q\rangle \in \mathbb{R}^2$, we have:

$$\text{curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

16.5.1 Green's Theorem (vector version)

Consider a line $\mathcal{C}$ such that $C = \partial D$, with $r(t) = (x(t), y(t))$ for $t \in [a, b]$. Note the following special vectors:

- Unit tangent vector:
$$\vec{T} = \frac{(x', y')}{\sqrt{(x')^2 + (y')^2}}$$
- Outward unit normal:
$$\vec{N} = \frac{(x', -y')}{\sqrt{(x')^2 + (y')^2}}$$
- Length element:
$$ds = \sqrt{(x')^2 + (y')^2}$$

Then we have that:
$$\oint_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \iint_D \mathrm{curl}(\vec{F})\, dA$$
is the *circulation* of $\vec{F}$ around $\mathcal{C}$. Likewise, we have that:
$$\oint_{\mathcal{C}} \vec{F} \cdot \vec{N}\, ds = \iint_D \mathrm{div}(\vec{F})\, dA$$
is the *flux* of $\vec{F}$ through $\mathcal{C}$.

16.6 Divergence & curl in 3D

Recall that in 2D, if we have $\vec{F} = \langle P, Q \rangle$, we had that $\mathrm{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$, and that $\mathrm{curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

In 3D, we have $\vec{F} = \langle P, Q, R \rangle$, and those operations are represented as follows:
$$\begin{aligned}\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right) \cdot \vec{F} \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

likewise,
$$\begin{aligned}\mathrm{curl}(\vec{F}) = \nabla \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}\end{aligned}$$

Interpreting the above quantities as representing the flow of an incompressible fluid, we can think of V as a small cube of fluid, and $\vec{F}$ as the velocity of V. Then $\text{div}(\vec{F}) = \frac{d}{dt}\text{vol}(V)$, and $\text{curl}(\vec{F}) = 2\omega_V$.

16.6.1 Identities

1. $\text{div}(\nabla f) = \nabla^2 f$ is the Laplacian of f.
2. $\text{curl}(\nabla f) = 0$.
3. $\text{div}(\text{curl} f) = 0$.

Finally, note the following two theorems:

Theorem 16.6.1. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. $\text{curl}(\vec{F}) = 0$ iff there exists a function f such that $\vec{F} = \nabla f$.

Theorem 16.6.2. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. $\text{div}(\vec{F}) = 0$ iff there exists a vector field $\vec{A}$ such that $\vec{F} = \text{curl}(\vec{A})$.

16.7 Parameterization of surfaces

Say that we have a parameterization of a surface in two variables, such that:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

We present several such examples of places where this parameterization can be found and used:

Example 16.7.1. Say that you have a function $f(x)$, which you would like to consider on $x \in [a, b]$, and rotate about the x-axis. Then, such a parameterization might be on x, θ, and look like:

$$\vec{r}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$$

where $(x, \theta) \in [a, b] \times [0, 2\pi]$.

Example 16.7.2. To parameterize a plane with normal vector $\vec{N}$, we find two vectors which span the plane (i.e., that their cross-product is in the direction of $\vec{N}$), as follows. Say, for example, that $\vec{N} = \langle 1, 1, 1 \rangle$. Then, we consider vectors $\vec{t} = \langle -1, 1, 0 \rangle$, and $\vec{u} = \langle 1, 0, -1 \rangle$, such that:

$$\vec{r}(u, v) = P + (\vec{u} \cdot \vec{t}) + (\vec{v} \cdot \vec{s})$$

Example 16.7.3. If we are given a function $f(x, y)$ and a region D, we can parameterize the surface of f over D naturally as:

$$\vec{r}(x, y) = \langle x, y, f(x, y) \rangle$$

Example 16.7.4. Finally, we consider the example of parametrizing a sphere S of radius a. We draw inspiration from our discussion of spherical coordinates which labels points with (ρ, φ, θ). We then compute the parameterization:

$$\vec{r}(\theta, \varphi) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\theta \rangle$$

16.8 Tangent planes

We conclude our discussion from the previous section with a brief discussion on finding tangent planes. If we are given (or can compute) a parameterization over the surface of some function, then we have $\vec{r}(u,v)$, where $\vec{r}_u$, and $\vec{r}_v$ are the instantaneous rates of changes along each parameterized direction.

Note crucially that:

$$\vec{r}_u \times \vec{r}_v = \frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0)$$

is normal to the tangent plane of S (i.e., the surface which is parameterized by $\vec{r}$) at (u_0, v_0).

Example 16.8.1. For surface S having parameterization:

$$\vec{r}(u,v) = \langle u^2, v^2, u + 2v \rangle$$

we wish to find the tangent plane at $(1,1,3)$. First, note that $u = v = 1$, and then we find that:

$$\begin{aligned} \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} \\ &= \langle -2v, -4u, 4uv \rangle \end{aligned}$$

then at $(1,1,3)$, $\vec{N} = (-2, -4, 4,)$, hence the plane is described by:

$$-2(x-1) - 4(x-1) + 4(x-3) = 0$$

In general, this process is described as follows:

1. If the surface S does not yet have a parameterization, parameterize it.
2. If the point on S does not have corresponding parameters, find them (i.e., such that $P = \vec{r}(u,v)$ for some given P).
3. Compute $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ by taking the component-wise partial derivatives with respect to each parameter.
4. Compute a vector in the direction of $\widehat{N}$, which is given as the cross product of the two partials found in the previous step.
5. Plug in the parameters to find the concrete value, and then use:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

to finish.

16.9 Surface area, integrals

Note that:

$$\iint_S z\, d\Sigma = \sum_{S_i \in S} \iint_{S_i} z\, d\Sigma_i, \qquad S = \dot{\bigcup}_i S_i$$

Example 16.9.1. Consider the lipstick tip described by three faces $S_1 : x^2 + y^2 = 1$, $S_2 : z = 0$, and $S_3 : z = 1 + x$, representing the side, bottom, and top, respectively.

1. S_1. Let $\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$, where $z \in [0, 1 + \cos\theta]$, and $\theta \in [0, 2\pi]$.

   Then:

   $$\vec{r}_\theta \times \vec{r}_z = \langle \cos\theta, \sin\theta, 0 \rangle$$

   and:

   $$\iint_{S_1} z\, d\Sigma = \int_0^{2\pi} \int_0^{1+\cos\theta} z \cdot |\vec{r}_\theta \times \vec{r}_z|\, dz d\theta = \cdots = 3\pi/2$$

2. S_2. Observe that:

   $$\iint_{S_2} z\, d\Sigma = \iint_{S_2} 0\, d\Sigma = 0$$

   holds trivially.

3. S_3. Finally, we consider the graph of $f(x, y) = 1 + x$ on the circle of radius 1. We have:

   $$\begin{aligned} \iint_{S_3} z\, d\Sigma &= \iint_D z\sqrt{f_x^2 + f_y^2 + 1}\, dxdy \\ &= \iint_D \sqrt{2}(1 + x)\, dxdy \\ &= \int_0^{2\pi} \int_0^1 \sqrt{2}(1 + r\cos\theta)\, r dr d\theta \\ &= \cdots = \sqrt{2}\pi \end{aligned}$$

and the result follows immediately by the disjointedness of the surface components..

A surface is *orientable* iff it has two sides. So, for e.g., the Möbius strip is not orientable. An *orientation* of a surface S is a continuous choice of normal vectors on S (either they point into S or out of S).

That is, for $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$:

$$\widehat{N} = \pm \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$$

Note that for orientable surfaces:

$$\iint_S \vec{F} \cdot d\Sigma = \iint_S \vec{F} \cdot \widehat{N}\, dS$$

is the flux of $\vec{F}$ across S, where:

$$\begin{aligned} d\Sigma &= \widehat{N} dS \\ &= \pm \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} \\ &= \pm(\vec{r}_u \times \vec{r}_v) du\, dv \end{aligned}$$

Example 16.9.2. Now consider $\vec{F} = \langle y, x, z\rangle$ with $S : z = 1 - (x^2 + y^2)$. On S, we have the trivial parameterization $\vec{r}(x, y) = \langle x, y, f(x, y)\rangle$, and thus:

$$\iint_{S_1} \vec{F} \cdot \vec{N}\, d\Sigma = \iint_D \vec{F}(\vec{r}_x \times \vec{r}_y)\, dA$$

where:

$$\begin{aligned} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) &= \langle P, Q, R\rangle \cdot \langle -f_x, -f_y, 1\rangle \\ &= \cdots = 4xy + 1 - (x^2 + y^2) \end{aligned}$$

so:

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot \vec{N}\, d\Sigma &= \iint_D (4xy + 1 - (x^2 + y^2))\, dA \\ &= \int_0^{2\pi} \int_0^1 (4r^2 \cos\theta \sin\theta + 1 - r^2)\, r dr d\theta = \cdots = \pi/2. \end{aligned}$$

16.10 Divergence Theorem

Recall that the surface integral

$$\iint_S \vec{F} \cdot \widehat{N}\, dS$$

computes the flux "out of S" in the field $\vec{F}$. Say, for example, that you are computing the flux out of the box with vertices at $(\pm 1, \pm 1, \pm 1)$. Because no surface captures this, one would have to compute:

$$\iint_E \vec{F} \cdot \widehat{N}\, dS = \sum_{i \in [6]} \iint_{E_i} \vec{F} \cdot \widehat{N}_i\, dS$$

that is, six integrals of flux, one for each face of the box around region E, and each with different normal vectors (suitable choices are $\pm\hat{\imath}$, and so on).

Instead of this, we can take advantage of the fact that E encloses a *region* in space, and express the surface integral as a volume integral over the divergence in the field using the Divergence Theorem:

Theorem 16.10.1 (Divergence Theorem)**.** Let S be a surface in space, and F a conservative vector field. Let E be the *region* enclosed by S. Then, the flux across the surface S in F is expressed equivalently as:

$$\iint_S \vec{F} \cdot \widehat{N}\, dS = \iiint_E \nabla \cdot \vec{F}\, dV$$

where E is the region enclosed by the surface S.

16.11 Stokes' Theorem

Theorem 16.11.1 (Stokes')**.** Let $\vec{F}$ be a field, and S a surface within the field whose boundary is defined by the closed curve C, where S is orientable along C. Then,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

To orient C along F, ensure that the "path along C" has the surface S to the *left* of the individual walking along the path. That is, $\widehat{N} \times \widehat{T}$ points *into* S.

We first consider an example from the lecture notes where it is easier to evaluate the line integral than the surface integral. In this case, we apply Stokes' theorem to rewrite the right-hand side as a line integral:

Example 16.11.1. Suppose that $\vec{F} = \langle xz, yz, xy \rangle$, and that S is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$.

First, we know from Stokes' theorem that:

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}.$$

Then, we must pick an appropriate parameterization of C. First, note that the intersection of the two regions occurs when $z = \sqrt{3}$. So, a parameterization of:

$$\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$$

suffices. The integral is then computed as follows:

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot d\vec{r} \\
&= \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} -\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t + 0 \, dt \\
&= \cdots = 0.
\end{aligned}$$

Next, we consider an example from the homework where it is easier to parameterize the surface in terms of $\vec{r}(u, v)$ and compute the line integral as a surface integral using Stokes' theorem.

Example 16.11.2. Suppose that $\vec{F} = \langle z^2, 2xy, 4y^2 \rangle$, and C is the path along which a point travels beginning at the origin, to $(2, 0, 0)$, then to $(2, 4, 1)$, then to $(0, 4, 1)$, and finally back to the origin. Find the work done by the particle along C through F.

Note that our goal is to answer solve for:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

where the right-hand side of the equality follows from Stokes' theorem. Note additionally that the right-hand side can be expressed as:

$$\begin{aligned}\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_S \nabla \times \vec{F} \cdot \widehat{N}\, dS \\ &= \iint_S \nabla \times \vec{F} \cdot (\pm(\vec{r}_u \times \vec{r}_v)\, dA)\end{aligned}$$

Note that S is the parallelogram spanned by $t = 2\hat{\imath}$, $s = 4\hat{\jmath} + \hat{k}$, so an appropriate parameterization is:

$$\begin{aligned}\vec{r}(u, v) &= u\langle 2, 0, 0\rangle + v\langle 0, 4, 1\rangle \\ &= \langle 2u, 4v, v\rangle\end{aligned}$$

and that $\vec{r}_u = 2\hat{\imath}$, as well as $\vec{r}_v = 4\hat{\jmath} + \hat{k}$. So,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 4 & 1 \end{vmatrix} = \langle 0, -2, 8\rangle$$

and:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2xy & 2y \end{vmatrix} = \langle 8y, 2z, 2y\rangle$$

Combining the two, we can evaluate the surface integral:

$$\begin{aligned}\iint_s \nabla \times F \cdot \widehat{N} dS &= \iint_S \langle 8y, 2z, 2y\rangle \cdot \langle 0, -2, 8\rangle\, dA \\ &= \iint_S (-4z + 16y)\, dA \\ &= \int_{v=0}^{1} \int_{u=0}^{1} 60v\, du dv = \cdots = 30.\end{aligned}$$