MATH 324 D, Braune

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Contents

Chapter 15

Multiple Integrals

15.1 Double Integrals over Rectangles

Recall that we approximate an integral of a function of a single variable by subdividing the region $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ with $\Delta x = (b - a)/n$, and approximate the integral as $n \to \infty$ as:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We can interpret the above Riemann summation over a single variable as analogous to determining the *area* below f, where f is a positive function. Likewise, we can take Riemann integrals over *areas* to determine volume.

If we let the region R be divided into $m \times n$ sub-rectangles, and define the area of each to be $\Delta A = \Delta x \Delta y$ (where $\Delta x = (b - a/n)$, and Δy is defined analogously), then we have that:

$$V \approx f(x_{ij}^*, y_{ij}^*)\Delta A$$

and that:

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

The quantity above stated on either side of the equality is the volume of the surface above the region R and below the surface f.

15.1.1 Midpoint Rule

One can approximate the value of a double integral by taking a pseudo-Riemann sum evaluated only at the midpoints of each sub-region:

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_{ij}, \overline{y}_{ij})\Delta A$$

15.1.2 Average Value

Recall that we had earlier for the integral over a function of one variable that:

$$\overline{f}_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Similarly, in the two-variable case, we have that:

$$\overline{f}_R = \frac{1}{A(R)} \iint_R f(x)\,dx$$

15.1.3 Properties of Double Integrals

Finally, note that double integrals are *linear*, meaning that they exhibit the following properties:

$$\iint_R (f(x,y) + g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

Likewise, when c is a constant, we have that:

$$\iint_R c\,f(x,y)\,dA = c \iint_R c\,f(x,y)\,dA$$

Finally, if $f \geq g$ for all $(x,y) \in R$, then:

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

15.1.4 Iterated Integrals

Consider $R = [a,b] \times [c,d]$, and observe that:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

When evaluating this integral, we:

- Work from the inside out, integrating f first with respect to a change in y, then with respect to a change in x.
- "Hold" the variables not being integrated as *constant* with respect to the variable currently being integrated against.

Example 15.1.1.

$$\begin{aligned}
\int_0^3 \int_1^2 x^2 y\,dy\,dx &= \int_0^3 \left[x^2 \frac{y^2}{2} \right]_{y=1}^2 dx \\
&= \int_0^3 x^2 \left[\frac{2^2 - 1^2}{2} \right] dx = \frac{3}{2} \int_0^3 x^2\,dx \\
&= \frac{3}{2} \left[\frac{x^3}{3} \right]_0^3 = \boxed{\frac{27}{2}}
\end{aligned}$$

Note that it is often convenient to change the *order* in which a function is integrated, and that this is a legal operation due to a theorem of Fubini.

Theorem 15.1.1 (Fubini's)**.** If f is continuous on a rectangle $R = [a,b] \times [c,d]$, then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

15.2 Double Integrals over General Regions

Call $D \subsetneq \mathbb{R}^2$ a non-rectangular region over which we would like to integrate $f(x,y)$. Then, we define:

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

if F is integrable over R (any rectangular region enclosing D), then we have that:

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

- Call D a *type I* region if D is defined:

  $$D = \{(x,y) : x \in [a,b], y \in [g_1(x), g_2(x)]\}$$

  where it is convenient to treat the nested integral over a single x_i, thus lending a natural form:

  $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

- Likewise, we call D a *type II* region if DA is defined

  $$D = \{(x,y) : y \in [c,d], x \in [h_1(y), h_2(y)]\}$$

  and it is likewise convenient to integrate it as

  $$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- If D is more complicated, then we subdivide it into regions D_i such that each D_i is itself either a type-I or type-II region and:

  $$D = \bigcup_i D_i, \quad \varnothing = D_i \cap D_j,\ i \neq j$$

  where the later statement means that any pairwise intersection of the sub-divided regions is non-overlapping. Therefore:

  $$\iint_D f(x,y)\,dA = \sum_i \int_{D_i} f(x,y)\,dA$$

Note the following two properties of double integrals: first, a constant dA integrated over D is the area of that region D:

$$\iint_D dA = A(D)$$

and that if $f(x,y) \in [m,M]$ for all $(x,y) \in D$, then:

$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$

15.3 Double Integrals in Polar Coordinates

Recall the polar coordinate system, which often makes it easier to integrate over circular or other complicated regions. Specifically, recall that:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \theta = \arctan(y/x)$$

a *polar rectangle* is thusly defined as:

$$R = \{(r,\theta) : r \in [a,b],\, \theta \in [\alpha,\beta]\}$$

and note that:

$$dA = r\,dr\,d\theta$$

15.4 Applications of Double Integrals

- To find the mass M of a planar object with density function $\rho : \mathbb{R}^2 \to \mathbb{R}$, note that:

$$M = \iint_D \rho(x,y)\,dA$$

- To find the average value of f in D, we have that:

$$\overline{f} = \frac{1}{A(D)} \iint_D f(x,y)\,dA$$

  and similarly to find the weighted average (according to the corresponding density function ρ), we have:

$$\overline{f} = \frac{1}{M(D)} \iint_D \rho(x,y) f(x,y)\,dA$$

- We also have that, for a planar object with density ρ, that the center of mass $(\overline{x},\overline{y})$ is given as:

$$\overline{x} = \frac{1}{M(D)} \iint_D x\rho(x,y)\,dA, \quad \overline{y} = \frac{1}{M(D)} \iint_D y\rho(x,y)\,dA$$

- Lastly, we can also compute the *moment of inertia* of an object spinning about some axis of rotation. Note that $\Delta m = \rho \Delta A$, and that $v = \omega r$. Then, we have that:

$$\begin{aligned} KE &= \frac{1}{2}(\Delta m)v^2 \\ &= \frac{1}{2}(\Delta m)(\omega r)^2 \\ &= \frac{1}{2}(\rho \Delta A)(\omega r)^2 = \frac{1}{2}(r^2 \rho \Delta A)\omega^2 \end{aligned}$$

  and that the moment of inertia about the origin is:

$$I_0 = \iint_D r^2 \rho \, dA$$

  Likewise, the moment of inertia about the x-axis is:

$$I_x = \iint_D y^2 \rho \, dA$$

  and finally, about the y-axis:

$$I_y = \iint_D x^2 \rho \, dA$$

15.5 Surface Area

Let S be the surface area of a solid with equation $z = f(x, y)$, where f has continuous partial derivatives with respect to the x-, y- and z-axes.

For simplicity, we assume that $f(x, y) \geq 0$, and that the domain D is over a rectangle. We use a Riemann-like process and divide D into sub-rectangle R_{ij}, and a corresponding point $P_{ij}(x_i, y_j, f(x_i, y_j))$ where $(x_i, y_j$ is the point in the region closest to the origin.

Taking the summation over all tangent planes, we arrive at:

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

where T_{ij} is the area of the tangent plane.

We must now find the area of some ΔT for a given i, j pairing. For small enough change in x, y, this portion of the surface represents a parallelogram in 3-dimensional space. It is uniquely defined by two perpendicular curves, which at infinitesimal zoom appear as vectors, not curves. Let us call these vectors $\vec{a}$ and $\vec{b}$. Note that:

$$\begin{aligned} \vec{a} &= (\Delta x, 0, f(x_0 + \Delta x, y_0) - f(x_0, y_0)) \\ &= \left(\Delta x, 0, \frac{\partial f}{\partial x}(x_0, y_0)\Delta x\right) \end{aligned}$$

$$\begin{aligned} \vec{b} &= (0, \Delta y, 0, f(x_0, y_0 + \Delta y) - f(x_0, y_0)) \\ &= \left(0, \Delta y, \frac{\partial f}{\partial y}(x_0, y_0)\Delta y\right) \end{aligned}$$

To compute the area of this parallelogram, we take the matrix determinant:

$$\begin{aligned} a \times b &\approx \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(x,y)\Delta x \\ 0 & \Delta y & f_y(x,y)\Delta y \end{vmatrix} \\ &= (-f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y) \end{aligned}$$

Then, the magnitude of $|\Delta T| = |\vec{a} \times \vec{b}|$:

$$\begin{aligned} |\Delta T| &\approx |\vec{a} \times \vec{b}| \\ &= \sqrt{f_x^2 + f_y^2 + 1}\, \Delta x \Delta y \end{aligned}$$

Integrating over the whole suface, we have:

$$\begin{aligned} A(S) &= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA \\ &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \end{aligned}$$

15.6 Triple Integrals

Whereas in the double integral case we integrated over a rectangular region (in the simple case), in the tripe integral case (by contrast) we integrate over $D \subsetneq \mathbb{R}^3$. A simple case is $D = [a,b] \times [c,d] \times [e,f]$.

Again, a Riemann-like argument shows that:

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{n=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) = \iiint_D f(x,y,z)\, dV$$

Note also that Fubini's theorem extends over triple integrals as well, meaning that dV can be broken down into any of the six combinations of $dx\, dy\, dz$.

We define three types of solid regions used to classify and evaluate triple integrals:

- A *type I* integral is given as $E = \{(x,y,z) : (x,y) \in D,\ z \in [u_1(x,y), u_2(x,y)]\}$ (that is, D is some region, and z varies between two heights.)

  In this instance, it is often convenient to evaluate it as:

  $$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

- A *type II* integral is given when x varies between two constants, but y varies as a function of x, and z as a function of x and y. In this instance, it is convenient to integrate it the region as:

  $$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

- Finally, a *type III* region is one where $(x,z) \in D$, and y varies as a function of x and z. It is typically convenient to integrate integrals of this form as:

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

15.6.1 Applications of Triple Integrals

- To find the *mass* occupied by some volume, we integrate the density of that space with respect to an infinitesimal change in volume:

$$m = \iiint_R \rho(x,y,z)\,dV$$

- Likewise, we can find the *moments* about the three coordinate axes as follows:

$$M_{yz} = \iiint_R x\rho(x,y,z)\,dV \qquad M_{xz} = \iiint_R y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_R z\rho(x,y,z)\,dV$$

- The above quantities can be used to determine the center of mass, which is given as: $(\overline{x},\overline{y},\overline{z})$:

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}$$

- Finally, if $\rho = c$ for all $(x,y,z) \in R$, then the center of mass is called the *centroid*. Likewise, the three moments of inertia about the coordinate axes are:

$$I_x = \iiint_R (y^2+z^2)\rho(x,y,z)\,dV \qquad I_y = \iiint_R (x^2+z^2)\rho(x,y,z)\,dV$$

$$I_z = \iiint_R (x^2+y^2)\rho(x,y,z)\,dV$$

15.7 Triple Integrals in Cylindrical Coordinates

In a cylindrical coordinate system, we make the following translation:

$$(x,y,z) \rightsquigarrow (r\cos\theta, r\sin\theta, z)$$

where:

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

Say we have a region E (of type I) which has projection D such that:

$$E = \{(x,y,z) : (x,y) \in D,\, z \in [u_1(x,y), u_2(x,y)]\}$$

and:

$$D = \{(r, \theta) : \theta \in [\alpha, \beta],\ r \in [h_1(\theta), h_2(\theta)]\}$$

We have that:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

or, converting to cylindrical coordinates, that:

$$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

15.8 Triple Integrals in Spherical Coordinates

Another useful coordinate system is the *spherical coordinate system*, which represents points (ρ, θ, ϕ), where $\rho = ||P||_2$, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the vector $\vec{P}$.
Note that:

- $\rho = c$ is a sphere with radius c.
- $\theta = c$ is the half-plane with angle c about the positive x-axis.
- $\phi = c$ is a half cone which is either above or below the xy-plane, depending on whether ϕ is less than or greater than $\pi/2$, respectively.

To convert between spherical and rectangular coordinates, observe the following relationships:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi$$

and that:

$$\rho^2 = x^2 + y^2 + z^2$$

Our "rectangular" unit in spherical coordinates is the spherical *wedge*, which is given as:

$$E = \{(\rho, \theta, \phi) : \rho \in [a, b],\ \theta \in [\alpha, \beta],\ \phi \in [c, d]\}$$

We divide E into smaller spherical wedges, E_{ijk}, by equally spaced spheres, half-planes, and half-cones. The volume is approximately a rectangular box, with $\Delta\rho$, $\rho_i\Delta\phi$, and $\rho_i \sin\phi_k \Delta\theta$. Thus, the approximate volume of E_{ijk} is given as:

$$\begin{aligned} \Delta V_{ijk} &\approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k \Delta\theta) \\ &= \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi \end{aligned}$$

An application of the Mean Value Theorem yields exact equality on ΔV_{ijk}, or that:

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k) \in E_{ijk}$. Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of this point. Then:

$$\begin{aligned}\iiint_E f(x,y,z)\,dV &= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk} \\ &= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi\end{aligned}$$

Hence,

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\rho_i^2 \sin\phi_k \, d\rho\, d\theta, d\phi$$

15.9 Change of Variables in Multiple Integrals

Recall that in single-variable calculus, we often perform a change of variable on a 1-dimensional integral as follows:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,\frac{dx}{du}\,du$$

More generally, we consider a transformation T from the uv-plane to the xy-plane as:

$$T(u,v) = (x,y)$$

where $x = g(u,v)$, and $y = h(u,v)$ (sometimes this is written as: $x = x(u,v)$, and $y = y(u,v)$). Note that we assume T is a C^1 *transformation*, meaning that g and h have continuous first-order partial derivatives. Note that the inverse of these transformations are denoted $u = G(x,y)$ and $v = H(x,y)$, respectively.
Say we have the image of S (some region in uv-space) projected onto the xy-plane. Note that we can write a function for the location of the position vector in the image as follows:

$$\vec{r}(u,v) = \langle g(u,v),\, h(u,v)\rangle$$

Note also that we can take the tangent vectors as the first-order derivatives of $\vec{r}$ as:

$$\vec{r}_u = \langle g_u(u_0,v_0),\, h_u(u_0,v_0)\rangle$$

and similarly:

$$\vec{r}_v = \langle g_v(u_0,v_0),\, h_v(u_0,v_0)\rangle$$

Then, we have a parallelogram whose edges are defined by $\vec{a}$, $\vec{b}$ as:

$$\begin{aligned}\vec{a} &= \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \\ \vec{b} &= \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)\end{aligned}$$

But, we have that:

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u}$$

so:

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \vec{r}_u$$

and:

$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \vec{r}_v$$

This means that we can take the Jacobian to determine the area ΔA. The transformation *Jacobian* is given as:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = [(\partial x/\partial u)(\partial y/\partial v)] - [(\partial x/\partial v)(\partial y/\partial u)]$$

Thus we have (from the previous calculation):

$$\Delta A \approx \left| \frac{\delta(x,y)}{\delta(u,v)} \right| \Delta u \, \Delta v$$

Note that:

$$\begin{aligned} \iint_D f(x,y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v \\ &= \iint_R f(g(u,v),\, h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \end{aligned}$$

In integrals of three variables, the situation is the same. Let T be a transformation map from S in uvw-space to R in xyz-space by:

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

Then, the Jacobian of T is given as:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

And thus that:

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w),\, y(u,v,w),\, z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$